Multiple Description Coding Over Many Channels

Raman Venkataramani
Harvard University, Cambridge, MA
raman@deas.harvard.edu

Workshop on Underwater Wireless Communications

April 8, 2003

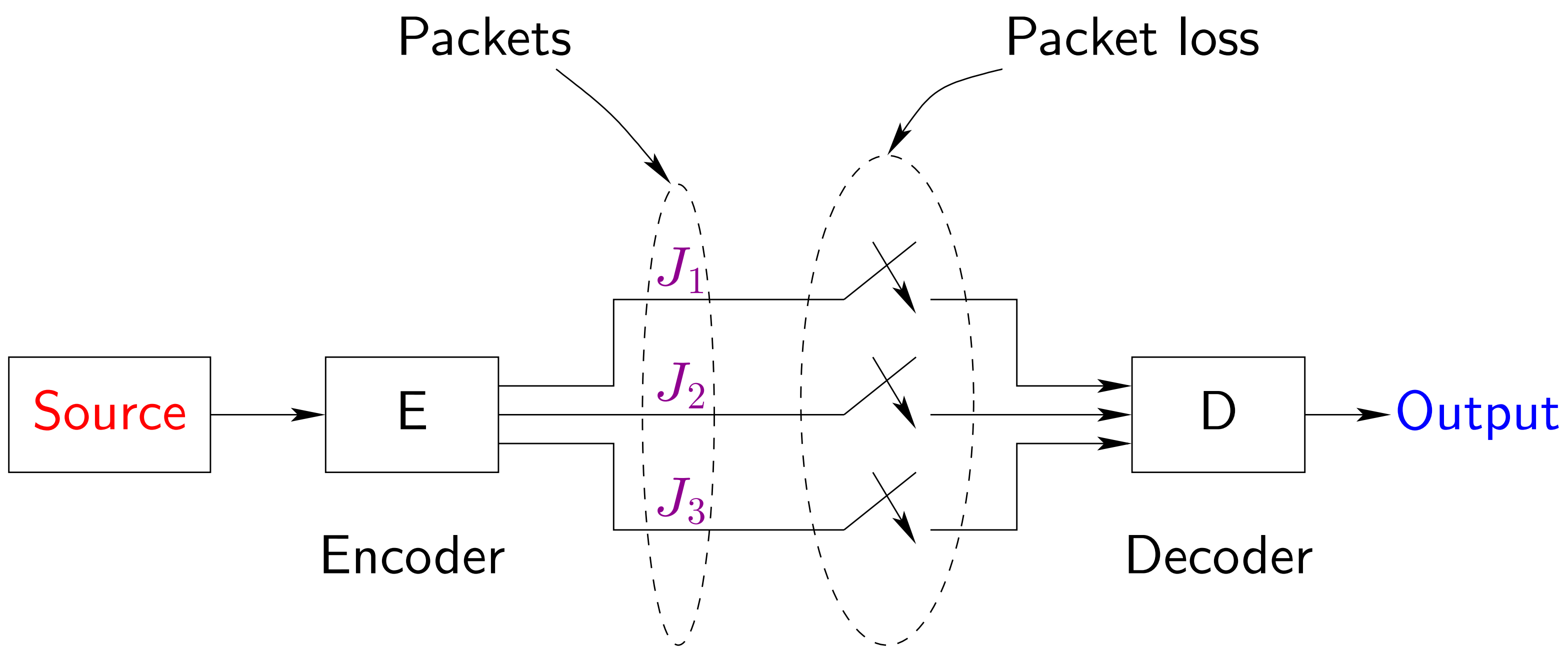
Outline

- Motivation for *Multiple Description Coding*
- Problem statement
- Review of known results (*Two-Description Coding*)
- New results for L-description coding ($L > 2$)
- Special cases and examples
- Summary

Underwater Wireless Channels

- Limited bandwidth and severe multipath propagation.
- Long delays (speed of sound in water is 1500m/s)
- Need for transmission of **real-time** data:
  - e.g., images or video data to/from divers.
- Poor propagation occasionally cause weak SNRs.

Erasure Channels



- Packets are either **lost completely** (erased) or **received error-free**

How do we deal with erasures?

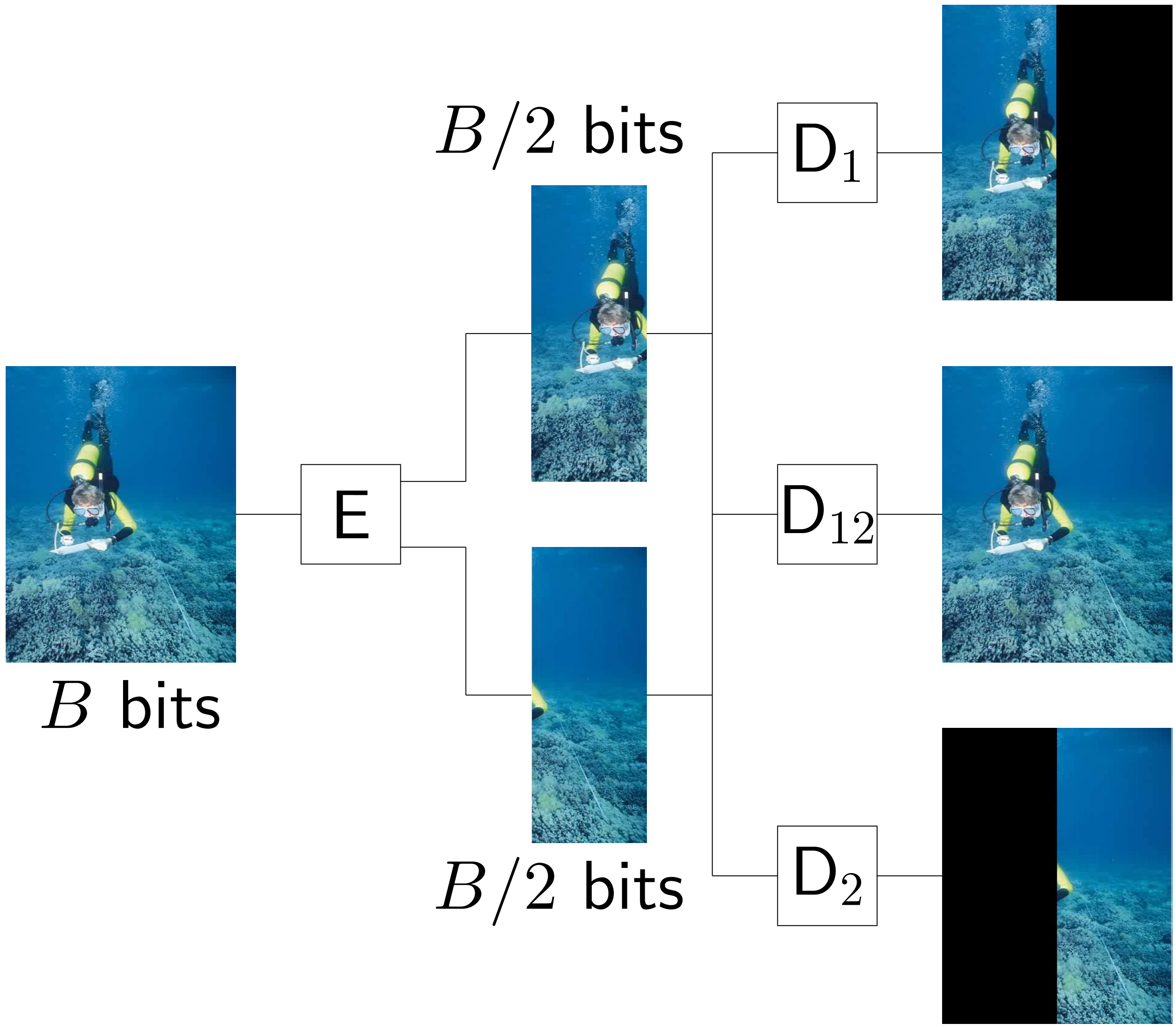
- Request a **retransmission**.
  - Ideal for loss-less transmission.
  - Not feasible for **real time** data such as **voice** and **video**.

Alternate Approach:

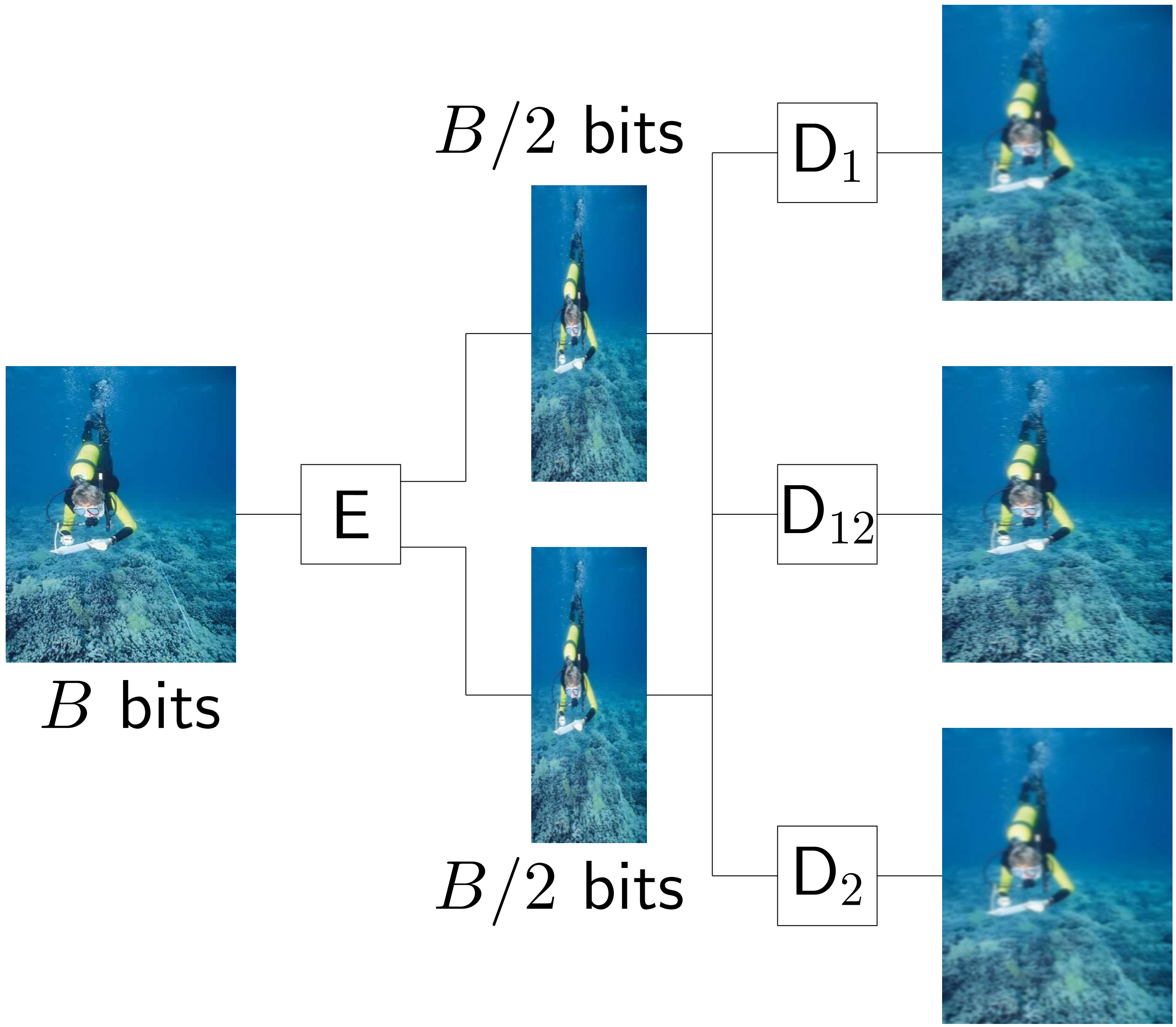
- Reconstruct using available packets
  - Requires adding **redundancy** to packets, i.e., **coding** across packets.

This approach is called **Multiple Description Coding**.

Transmitting Independent Packets



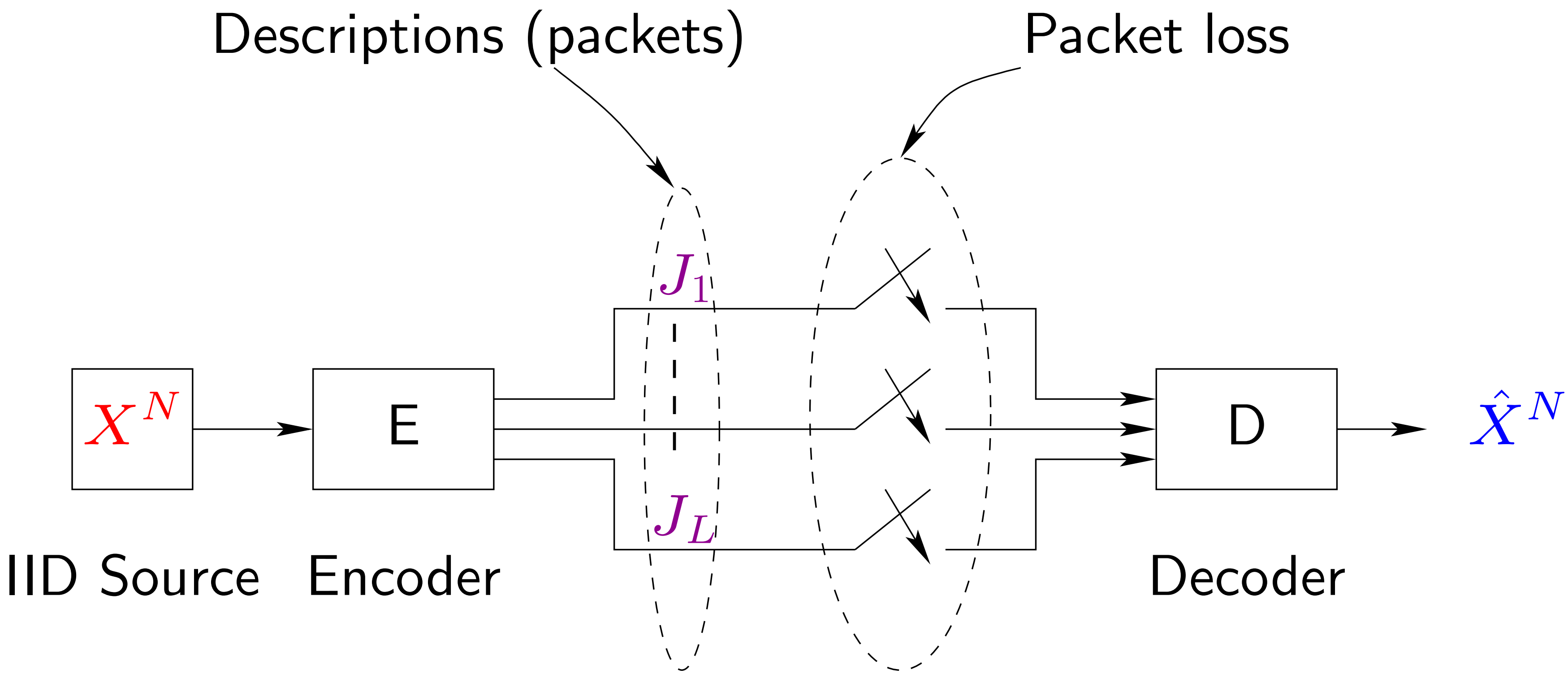
Transmitting Correlated Packets



Multiple Description (MD) Coding vs. Compression

- **Compression** (no coding across packets):
  - **Best reconstruction** when all packets are received.
  - Sharp degradation as packet losses increase.
- **Multiple Description Coding**:
  - **Graceful degradation** as packet losses increase.
  - Suboptimal when all packets are received.

Multiple Description Coding



Source: Length N vector X^N of i.i.d. random variables.

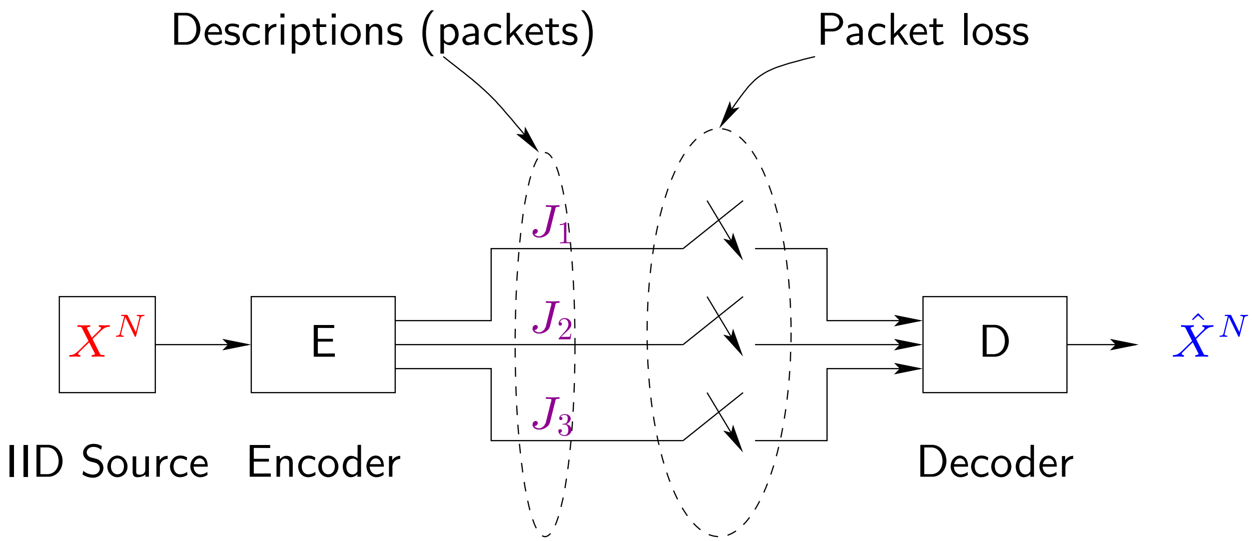
Encoder: $X^N \to \{J_1, \ldots, J_L\}$ which are the L "descriptions" of X^N at rates R_1,. . . R_L per source symbol.

Descriptions:
$J_l = f_l(X^N), \quad H(J_l) \leq NR_l, \quad l = 1, \ldots L.$

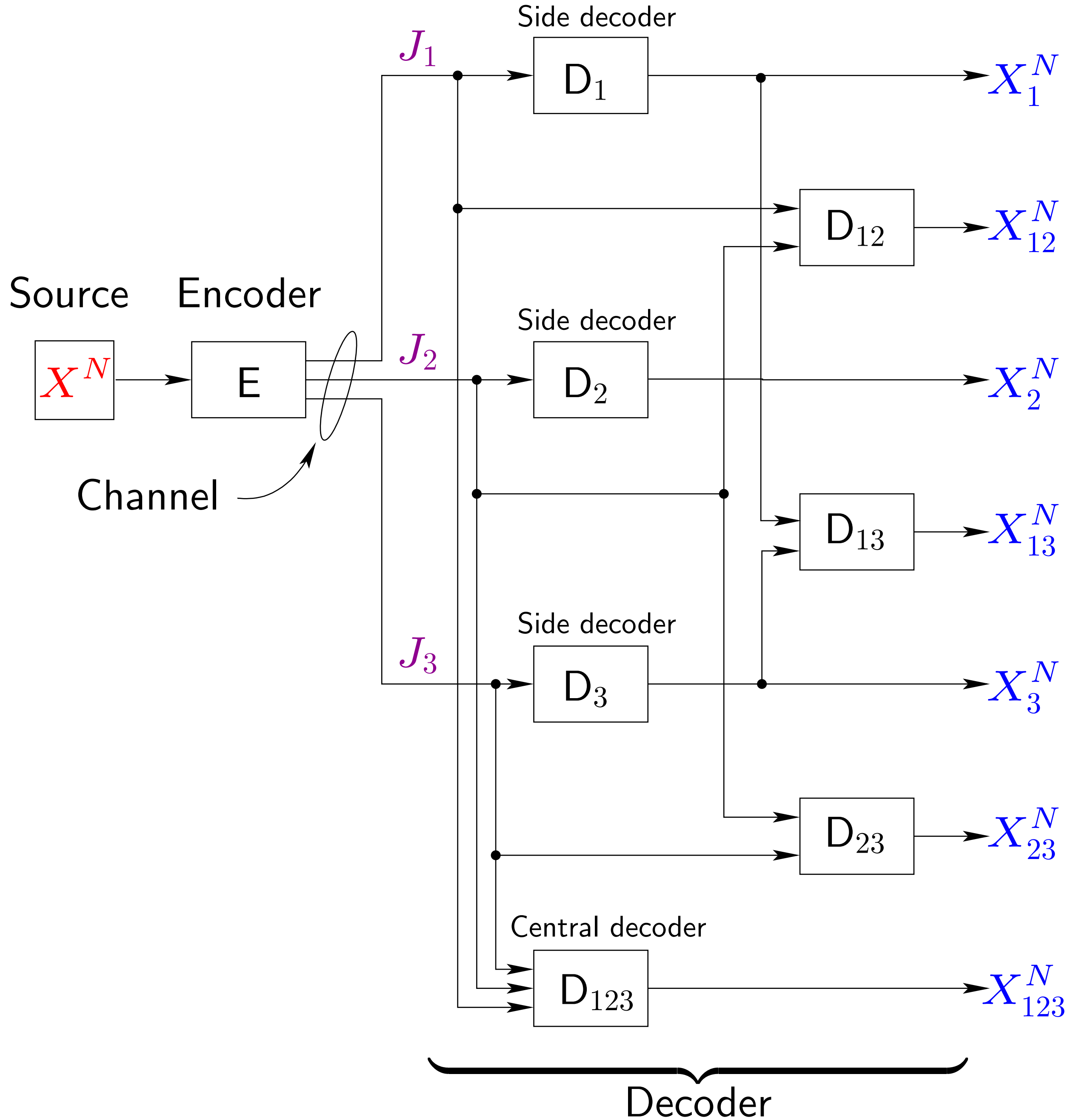
Decoder: Consists of $2^L - 1$ sub-decoders: one for each non-empty subset of the available descriptions.

Decoder Outputs: $X^N_{\mathcal{S}} = g_{\mathcal{S}}(\{J_l : l \in \mathcal{S}\})$ where $\mathcal{S} \subseteq \{1, \ldots, L\}$, $\mathcal{S} \neq \emptyset$.

Example: Coding with 3 Descriptions



- The source is i.i.d. vector X^N (length N).
- The encoder outputs L *descriptions* J_1, J_2, J_3 of X^N.
- The decoder produces an output $\hat{X}^N$ from the **available** descriptions.


Source
Encoder
X^N
E
Channel
J_1
J_2
J_3
Side decoder
D_1
Side decoder
D_2
Side decoder
D_3
D_{12}
D_{13}
D_{23}
Central decoder
D_{123}
X_1^N
X_{12}^N
X_2^N
X_{13}^N
X_3^N
X_{23}^N
X_{123}^N
Decoder

Problem Statement

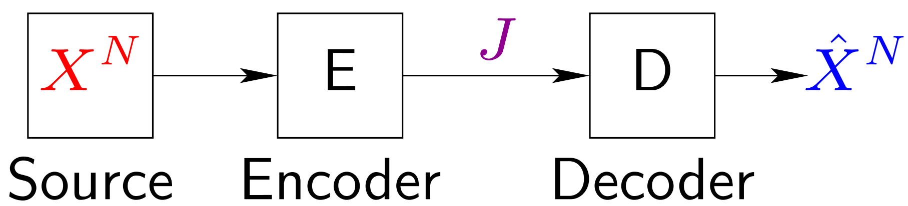
- **Problem**: **What is the Rate-Distortion (R-D) region?**
- The **rates** (L parameters) are $R_1, \ldots, R_L$.
- The **distortions** ($(2^L - 1)$ parameters) are

$$D_{\mathcal{S}} = \frac{1}{N}\mathrm{E}d(X^N, X_{\mathcal{S}}^N), \quad \mathcal{S} \subseteq \{1, \ldots, L\}, \mathcal{S} \neq \emptyset$$

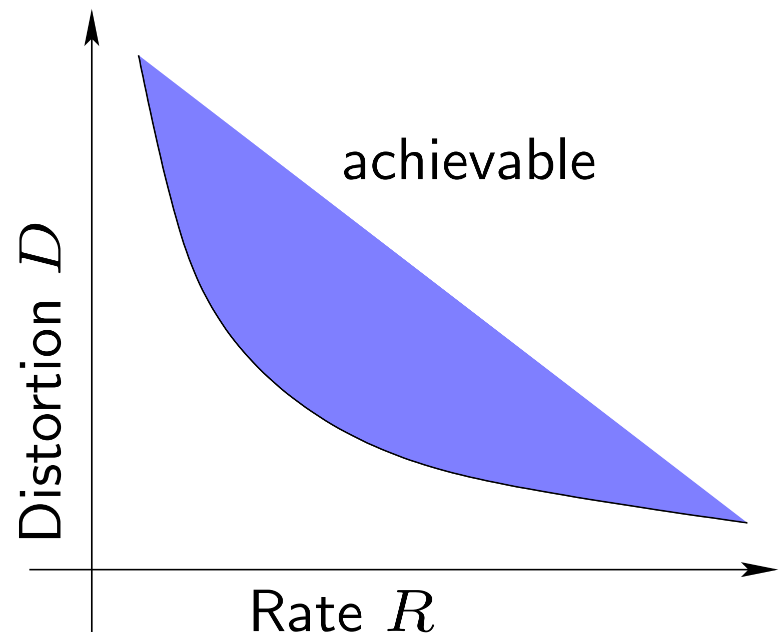
- The R-D region is $(L + 2^L - 1)$-dimensional.

- **Why do we need the R-D region?**
  - Provides complete knowledge about tradeoff between packet redundancy and fidelity of reconstruction.
  - Helps design good MD coders.
- For $L = 1$, it is Shannon's R-D region.

Rate-Distortion Theory



$$\frac{1}{N} H(J) \le R$$
$$\frac{1}{N} d(X^N, \hat{X}^N) \le D$$



The R-D region is the set of achievable (R, D) as $N \to \infty$.

Theorem 1. [Shannon] *The R-D region is the convex set* $R \geq R(D)$ *where*

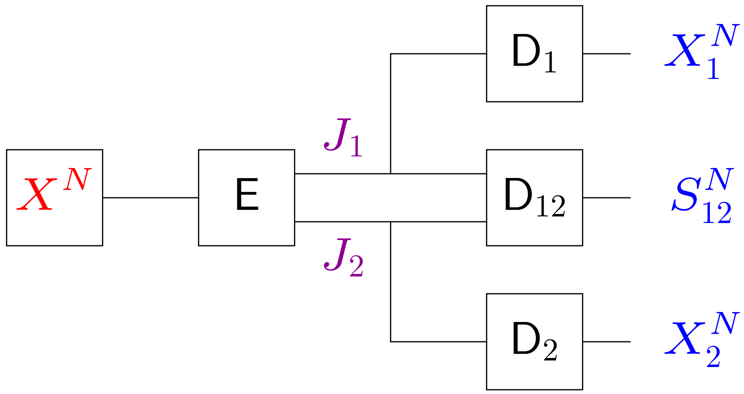
$$R(D) = \min_{\hat{X}} I(X; \hat{X}) \quad \textit{s.t.} \quad \mathrm{E}d(X, \hat{X}) \leq D$$

minimized over all $\hat{X}$ *jointly distributed with* X*.*

For the **Gaussian Source**: $X \sim N(0, 1)$, the **rate-distortion** fucntion is

$$R(D) = \frac{1}{2} \log \left(\frac{1}{D} \right)$$

Two-Description Coding



The rates and distortions are:

$$R_1 = \frac{1}{N}H(J_1) \qquad D_1 = \frac{1}{N}\mathrm{E}d(X^N, X_1^N)$$

$$R_2 = \frac{1}{N}H(J_1) \qquad D_2 = \frac{1}{N}\mathrm{E}d(X^N, X_2^N)$$

$$D_{12} = \frac{1}{N}\mathrm{E}d(X^N, X_{12}^N)$$

Two-Description Coding

El Gamal and Cover (1982) found an **achievable region** for $L = 2$:

$$R_1 \geq I(X; X_1)$$

$$R_2 \geq I(X; X_2)$$

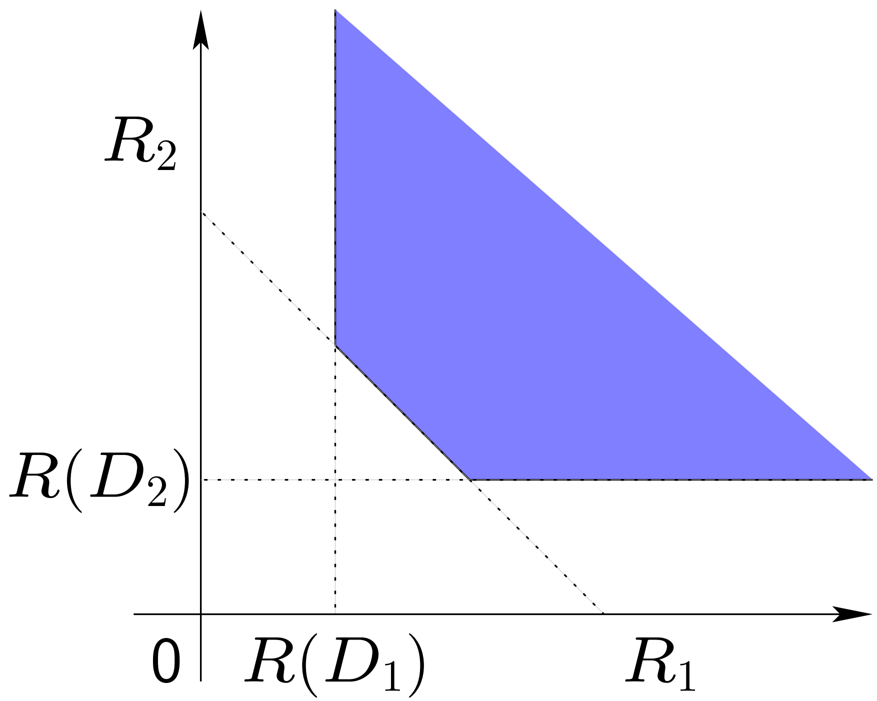
$$R_1 + R_2 \geq I(X; X_1 X_2 X_{12}) + I(X_1; X_2)$$

$$D_{\mathcal{S}} \geq \mathrm{E}d(X, X_{\mathcal{S}}), \quad \mathcal{S} = 1, 2, 12$$

where X_1, X_2, X_{12} are any r.v's jointly distributed with the source X.

- The convex hull of this region is achievable by time-sharing.
- This is *an* achievable region, **not necessarily** $\boldsymbol{the}$ **R-D region**.
- **Ozarow (1980)** computed the R-D region for the Gaussian Source.

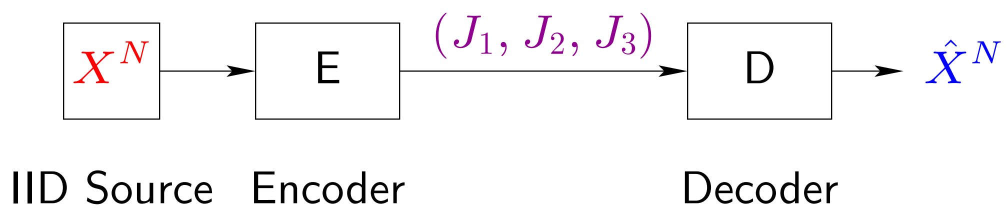
The Gaussian Source



The Gaussian rate region

- **Ahlswede (1987)** showed tightness of the inner bound by El Gamal and Cover in the “no excess rate” case ($R_1 + R_2 = R(D_{12})$) for the joint description output: X_{12}.
- **Zhang and Berger (1987)** provided a stronger achievable result than El Gamal and Cover for $L = 2$. For the **binary symmetric source** with Hamming distortion measure, their result provides a **strict improvement**.

Progressive Data Transmission



The decoding is **progressive**:

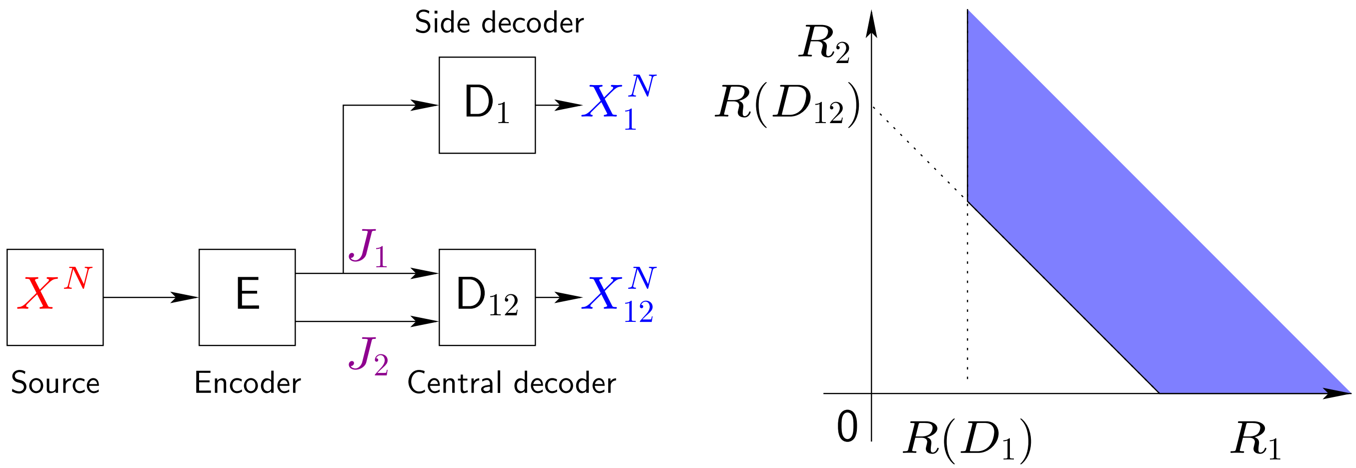
$$(J_1) \to D_1 \geq D(R_1)$$
$$(J_1, J_2) \to D_{12} \geq D(R_1 + R_2)$$
$$(J_1, J_2, J_3) \to D_{123} \geq D(R_1 + R_2 + R_3)$$

- **Question**: Can we attain all the minimum distortions simultaneously?
- **Answer**: Not always. X is **successively refinable** if the Shannon bounds are achievable with equality:

$$D_1 = D(R_1)$$
$$D_{12} = D(R_1 + R_2)$$
$$D_{123} = D(R_1 + R_2 + R_3)$$

Successive Refinement

- The Gaussian source is successively refinable:



Rate region for fixed D_1 and D_{12}

R-D region: $R_1 \geq R(D_1), \quad R_1 + R_2 \geq R(D_{12}).$

- This structure is a **special case of MD coding** with X_2 absent.

Summary of New Results

- Achievable region for MD coding of an arbitrary source ($L \geq 2$).
- R-D region bounds for the Gaussian source.
- Special cases of Gaussian MD coding.
- Examples.

Achievable Region for $L \geq 2$

Theorem 2. *The R-D region contains the rates and distortions satisfying*

$$\sum_{l \in \mathcal{S}} R_l \geq (|\mathcal{S}|-1) I(X; X_\emptyset) - H(X_\mathcal{U} : \mathcal{U} \in 2^\mathcal{S} | X)$$
$$+ \sum_{\mathcal{T} \subseteq \mathcal{S}} H(X_\mathcal{T} | X_\mathcal{U} : \mathcal{U} \in 2^\mathcal{T} - \mathcal{T})$$
$$D_\mathcal{S} \geq \mathrm{E} d_\mathcal{S}(X, X_\mathcal{S})$$

for every $\emptyset \neq \mathcal{S} \subseteq \mathcal{L} = \{1, \ldots, L\}$ *and some joint distribution between outputs* $\{X_\mathcal{S}\}$ *and the source* X*.*

- Best region is obtained by **optimizing over the joint p.d.f** of $(X, \{X_{\mathcal{S}} : \mathcal{S} \in 2^{\mathcal{L}}\})$.
- For a given p.d.f, the rate-region is a polyhedron:
  – Every constraint is active (polymatroid structure).
- Region is an inner bound on the R-D region.
- Proof uses random coding.
- For $L = 2$, this produces the results of **El Gamal and Cover**, and of **Zhang and Berger**.

Outer Bound on the R-D Region

- Outer bounds: difficult to find for **arbitrary sources**.
  - Proof depends strongly on properties of source.
- We focus on
  - **Gaussian source**: $X \sim N(0,1)$.
  - **Squared-error distortion**: $d(x, y) = |x - y|^2$.

- An **outer bound** on the R-D region:

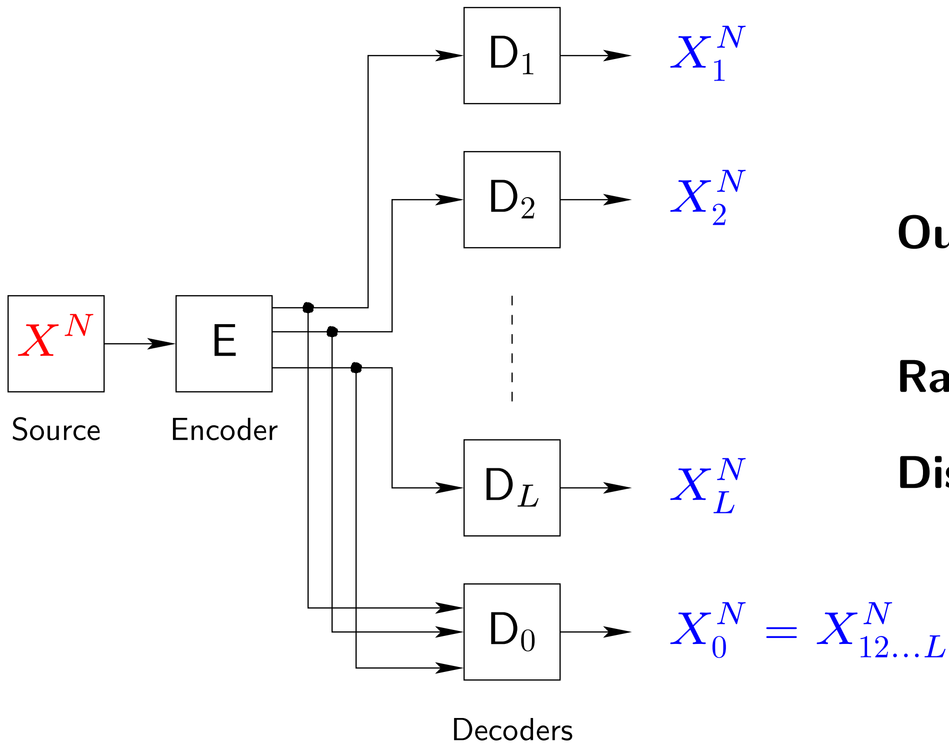
Theorem 3. *The R-D region is contained in*

$$\sum_{k\in\mathcal{K}} R_k \geq -\frac{1}{2}\log\left[\min_{\{\mathcal{K}_m\}_{m=1}^{M}} \inf_{\lambda\geq 0}\left(D_{\mathcal{K}}\frac{\prod_{m=1}^{M}(D_{\mathcal{K}_m}+\lambda)}{(D_{\mathcal{K}}+\lambda)(1+\lambda)^{M-1}}\right)\right], \quad \forall\mathcal{K}\in 2^{\mathcal{L}}$$

minimized over all partitions $\{\mathcal{K}_m\}$ *of* $\mathcal{K}$*.*

- The outer bound is a polyhedron.
- We also have a stronger outer bound.

L Channels and $L+1$ Decoders



Outputs: X_1^N, X_2^N, . . . , X_L^N and $X_0^N = X_{12\ldots L}^N$

Rates: R_1, R_2, . . . , R_L

Distortions: D_1, D_2, . . . , D_L and D_0

Keep only **side** and **central** decoders. Ignore all other decoders.

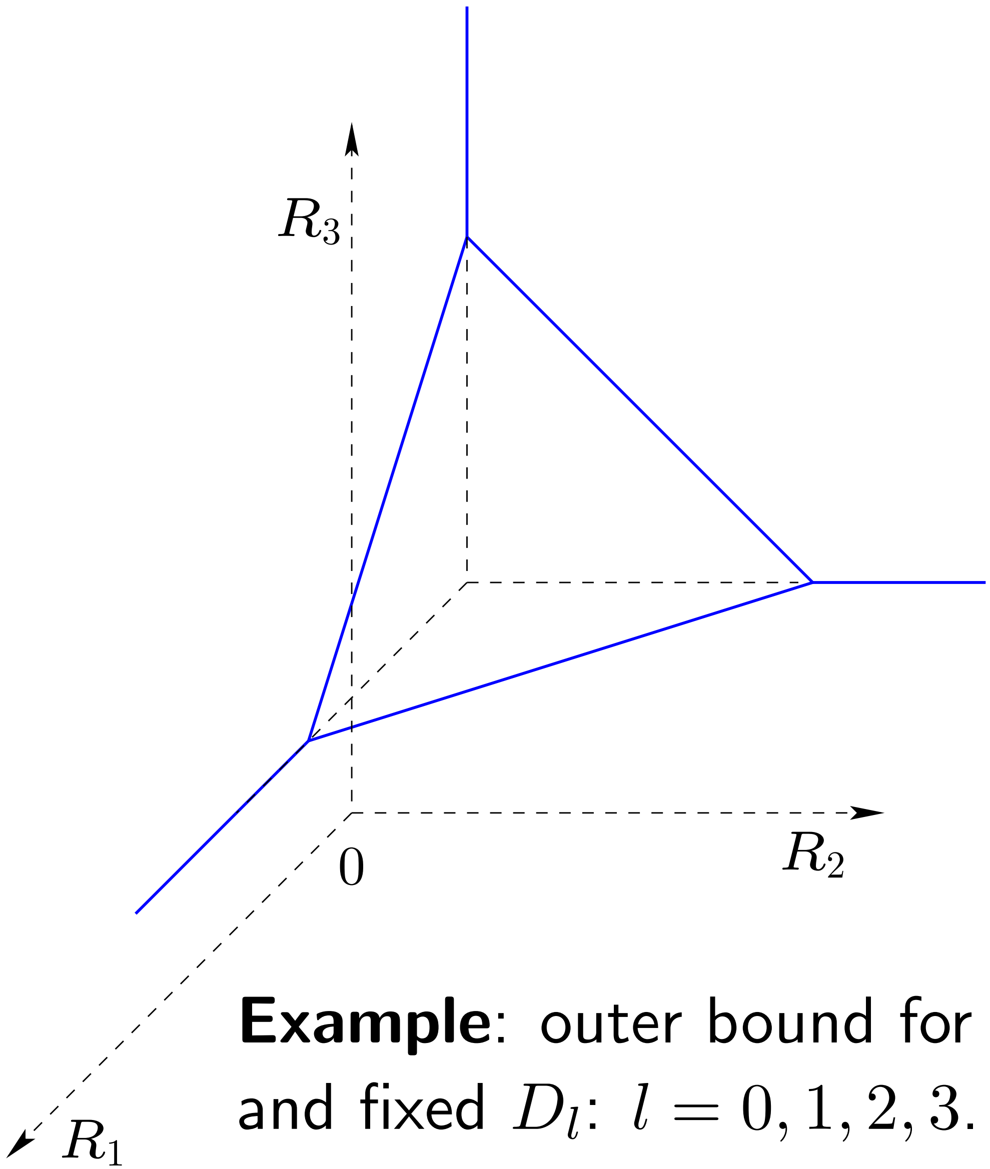
Bounds on the R-D region

- **Outer Bound**: For the L-channel $(L+1)$-receiver problem, the *outer bound* reduces to

$$R_l \geq -\frac{1}{2}\log D_l, \quad (l = 1, \ldots, L)$$

$$R_1 + \cdots + R_L \geq -\frac{1}{2}\log \varphi_L(D_0, \ldots, D_L)$$

$$\varphi_L(D_0, \ldots, D_L) = \inf_{\lambda \geq 0}\left[D_0 \frac{\prod(D_l + \lambda)}{(D_0 + \lambda)(1 + \lambda)^{L-1}}\right]$$



Example: outer bound for $L = 3$ and fixed D_l: $l = 0, 1, 2, 3$.

Inner Bound

Pick a **joint Gaussian p.d.f.** for $(X, \{X_{\mathcal{S}} : \mathcal{S} \in 2^{\mathcal{L}}\})$:

Let $(W_1, \ldots, W_L)$ be jointly Gaussian with covariance matrix $\boldsymbol{K}$.

$$X_l = \mathrm{E}(X | X + W_l) \equiv \alpha_l (X + W_l), \quad (l = 1, \ldots, L)$$

$$X_0 = \mathrm{E}(X | X_1, \ldots, X_L) \equiv \sum_{l=1}^{L} \beta_l X_l$$

for appropriate α_l, β_l.

- For given distortions D_1,. . . , D_L and D_0, we optimize over $\boldsymbol{K}$ to find the best rate region.
- The optimal solution takes the form:

$$\boldsymbol{K} = \begin{pmatrix} \sigma_1^2 & u & u & \dots & u \\ u & \sigma_2^2 & u & \dots & u \\ u & u & \sigma_3^2 & \dots & u \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u & u & u & \dots & \sigma_L^2 \end{pmatrix}$$

Tightness of Bounds

The inner and outer bounds meet for some rates and distortions.

Theorem 4. *For any* $L \geq 2$ *and distortions* $D_0, D_1, \ldots, D_L$*, the best achievable* $\mathbf{sum\ rate}$ *meets the outer bound:*

$$\sum_{l=1}^{L} R_l = 1\frac{1}{2}\log \varphi_L(D_0, \ldots, D_L).$$

Example: 3-Channel 4-Decoder Problem

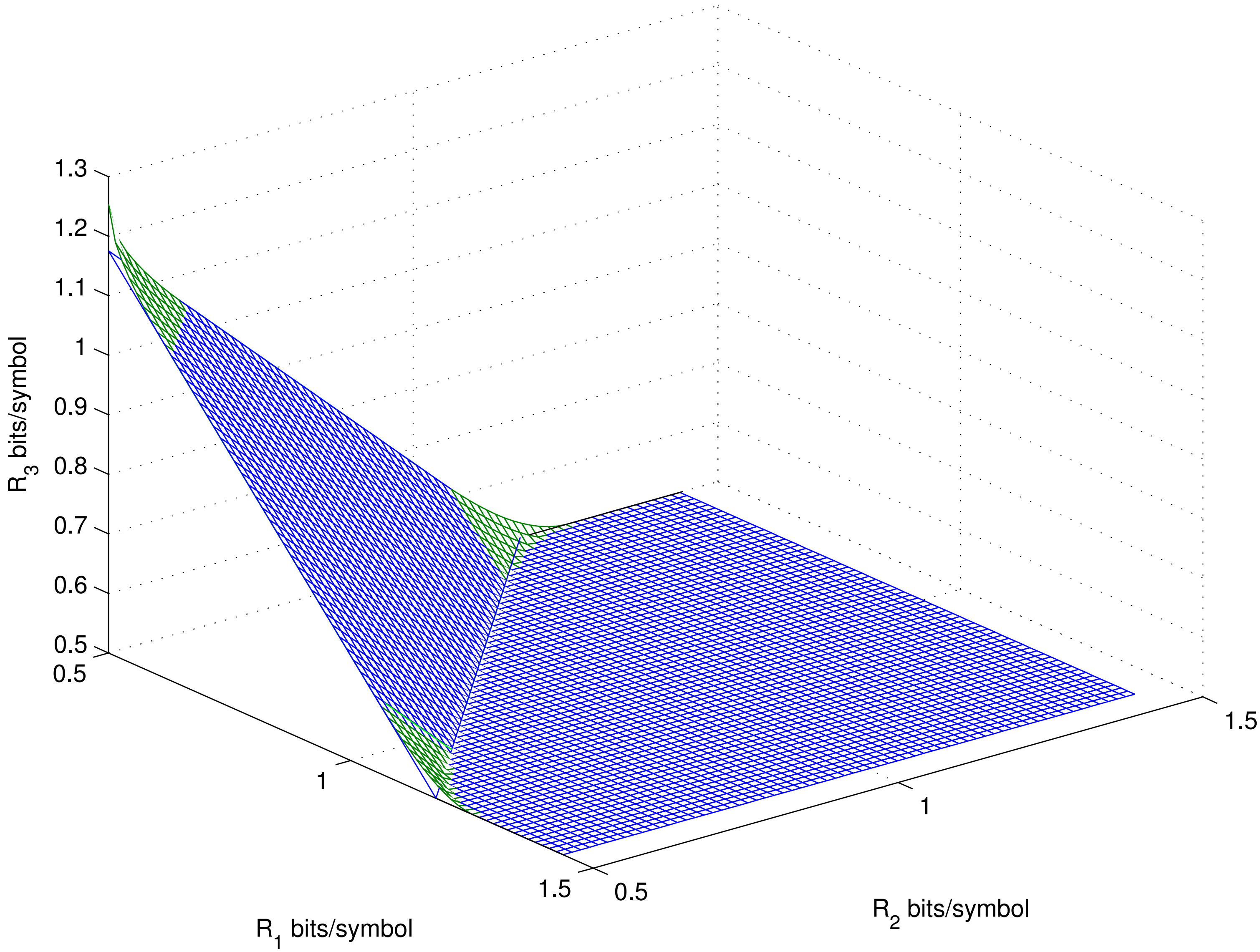
Take $L = 3$, $D_1 = D_2 = D_3 = 1/2$ and $D_0 = 1/16$.

Outer Bound:

$$R_l \geq 0.5, \quad l = 1, 2, 3$$
$$R_1 + R_2 + R_3 \geq 2.1755$$

Achievable Rates:

$$R_l \geq 0.5, \quad l = 1, 2, 3$$
$$R_1 + R_2 + R_3 = 2.1755$$
$$R_l + R_m \geq 1.1258, \quad l < m$$



Blue Region: Inner and outer bounds meet on a hexagon.

Green Region: Inner bound (does not meet outer bound).

Theorem 5. *Suppose*

$$D_L + \left(1 - L + \sum_{l=1}^{L-1} D_l^{-1}\right)^{-1} \geq 1 + D_0.$$

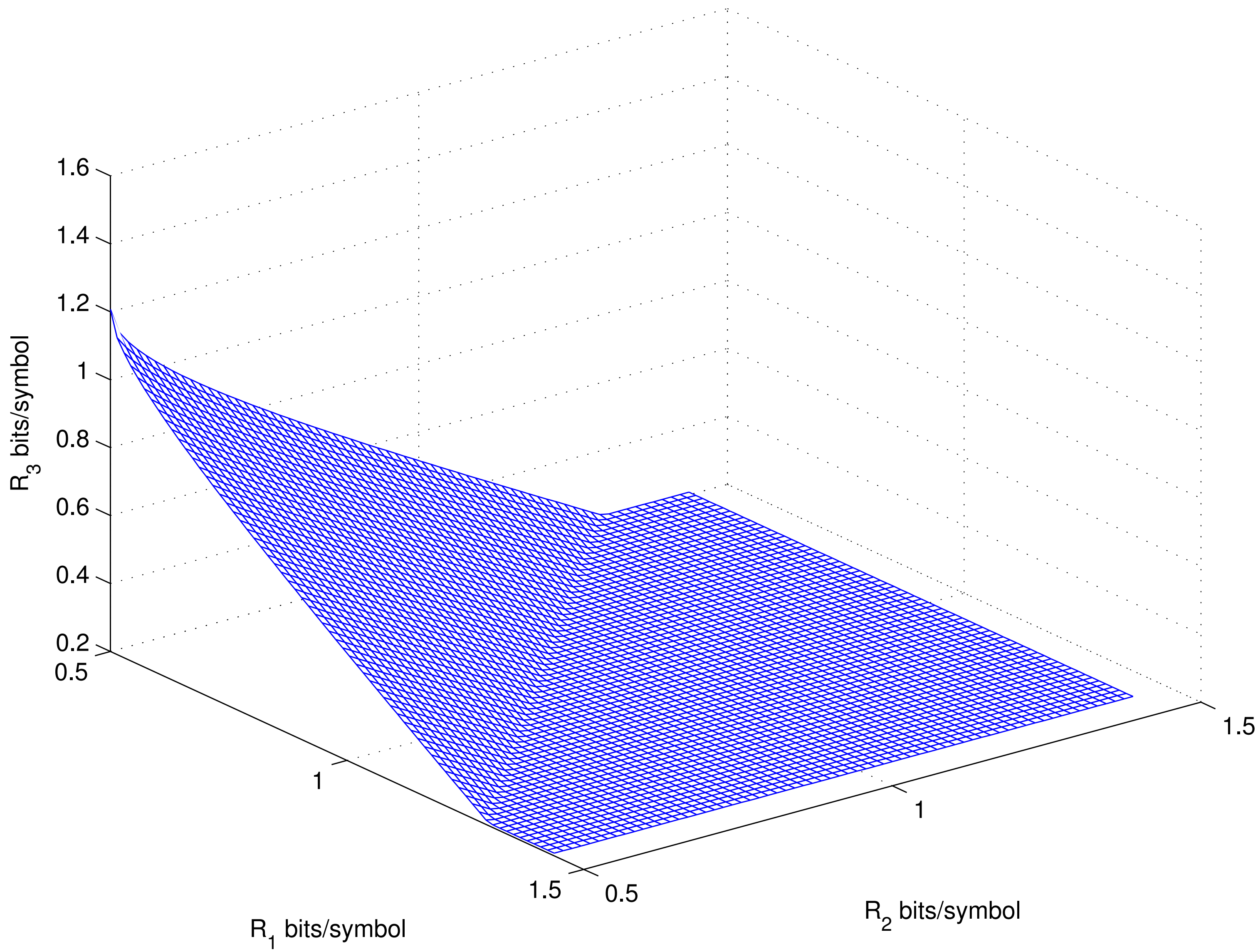
then the outer and inner bounds meet everywhere.

Example: $L = 3$, $D_1 = D_2 = 1/2$, $D_3 = 3/4$, and $D_0 = 1/16$:

The above inequality

$$D_3 + \frac{D_1 D_2}{D_1 + D_2 - D_1 D_2} \geq 1 + D_0$$

is satisfied.



The R-D Region

Successive Refinement on Chains

$$X \xrightarrow{R_1} X_1 \xrightarrow{R_2} X_{12} \xrightarrow{R_3} X_{123} \ldots$$

Gaussian

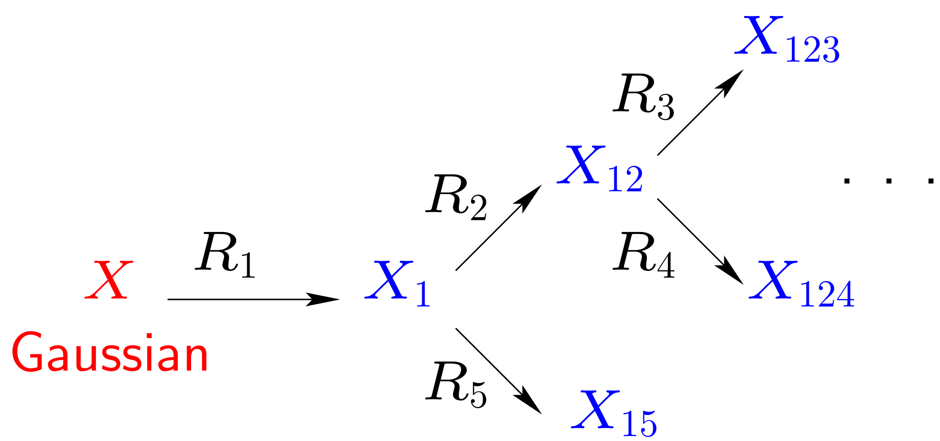
$$D_1 \geq D(R_1)$$

$$D_{12} \geq D(R_1 + R_2)$$

$$D_{123} \geq D(R_1 + R_2 + R_3)$$

Proved using the achievability result.

Successive Refinement on Trees



$$D_1 \geq D(R_1)$$

$$D_{12} \geq D(R_1 + R_2)$$

$$D_{123} \geq D(R_1 + R_2 + R_3)$$

$$D_{124} \geq D(R_1 + R_2 + R_4)$$

$$D_{15} \geq D(R_1 + R_5)$$

Summary

- **Multiple Description Coding**:
  - Discussed applications to real-time transmission over underwater acoustic networks.
  - Reviewed known MD coding results.
- **Our new results**:
  - Achievable rate region for MDC with *many* channels.
  - Outer bound on the R-D region for the Gaussian source.
  - Demonstrated tightness of bounds.