Sampling Theorems for Uniform and Periodic Nonuniform MIMO Sampling of Multiband Signals

Raman Venkataramani, *Member, IEEE*, and Yoram Bresler, *Fellow, IEEE*

***Abstract*—We examine a multiple-input multiple-output (MIMO) sampling scheme for a linear time-invariant continuous-time MIMO channel. The input signals are modeled as multiband signals with different spectral supports, and the channel outputs are sampled on either uniform or periodic nonuniform sampling sets, with possibly different but commensurate intervals on the different outputs. This scheme encompasses Papoulis' generalized sampling and several nonuniform sampling schemes as special cases. We derive necessary and sufficient conditions on the channel and the sampling rate that allow stable perfect reconstruction of the inputs or, equivalently, perfect inversion of the channel. From an implementation viewpoint, we note that it is desirable that the reconstruction filters have continuous frequency responses. We derive necessary and sufficient conditions that guarantee this continuity property. The frequency responses of the reconstruction filters are specified as solutions to a system of linear equations. Finally, we demonstrate that perfect reconstruction may be possible, even when the channel outputs are sampled at an average rate that does not allow the reconstruction of any output from its samples alone. In certain instances, this average rate can achieve the recently presented fundamental bounds on MIMO sampling density.**

***Index Terms*—Interpolation, MIMO equalization, minimum-rate sampling, multiband signals, multichannel deconvolution, multiple-input-multiple-output channel, reconstruction, source separation, stable sampling.**

I. Introduction

The study of multiple-input multiple-output (MIMO) channel equalization is motivated by applications in multichannel deconvolution and multiple source separation. Some example applications where MIMO channels arise are multiuser or multiaccess wireless communications and space-time coding with antenna arrays or telephone digital subscriber loops [1]–[4], multisensor biomedical signals [5], [6], multitrack magnetic recording [7], multiple speaker (or other acoustic source) separation with microphone arrays [8], [9], geophysical data processing [10], and multichannel image restoration [11], [12].

In practice, the MIMO channel equalizer is implemented using digital signal processors. However, the channel inputs and outputs are continuous-time signals, implying that the channel outputs need to be sampled prior to processing by the digital system. Hence, the problem is equivalent to reconstructing the channel inputs from the sampled output signals. In other words, the MIMO channel inversion problem can be restated as one in sampling theory, and we call this sampling scheme *MIMO sampling*.

To focus on the sampling issues, we restrict our attention in this paper to the scenario of a linear time-invariant continuous-time MIMO channel with known frequency response matrix. The harder problem of sampling conditions for blind channel inversion is left for future work.

The study of MIMO sampling has useful practical implications. Most work to date on multichannel deconvolution has addressed discrete-time channel models, apparently assuming that each output is sampled at the appropriate *Nyquist rate* sufficient for reconstruction of each output. Here, Nyquist rate is defined as the smallest uniform sampling rate that guarantees no aliasing. i.e., no overlap of the signal's spectral support with its translates by multiples of the sampling frequency. However, as we demonstrate in this paper, this is not necessary, and appropriately chosen uniform or nonuniform sampling schemes with lower average sampling density can suffice for perfect reconstruction of the MIMO channel inputs.

Although motivated by real-world problems, MIMO sampling is an important problem in sampling theory in its own right. Several sampling schemes can be expressed as special cases of the MIMO setting. For example, for a single-input multiple-output (SIMO) channel, the outputs are filtered and uniformly sampled versions of a single input signal. In other words, this is precisely Papoulis' generalized sampling [13]. Additionally, if the channel filters are chosen to be pure delays, one obtains multicoset or periodic nonuniform sampling of the input which has been widely studied [14]–[25], as it allows us to approach the Landau minimum sampling rate for multiband signals [26]. Seidner and Feder [27] provide a natural generalization of Papoulis' sampling expansions for a vector inputs whose components are bandlimited to $[-B, B]$. Their sampling scheme is clearly a special case of MIMO sampling. We deal only with multiband signal spaces; see [28] for some results on multichannel sampling for general signal spaces such as wavelet and spline spaces.

Fig. 1 is the block diagram for MIMO sampling. The channel is shown to the left of the dashed line, and its inputs $x_r(t)$ are

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R. Venkataramani was with the Division of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138 USA. He is now with Seagate Technology, Pittsburgh, PA 15222 USA (e-mail: ramanv@ieee.org).

Y. Bresler is with the Coordinated Science Laboratory, Department of Electrical and Computer Engineering, University of Illinois, Urbana, IL 61801 USA (e-mail: ybresler@uiuc.edu).

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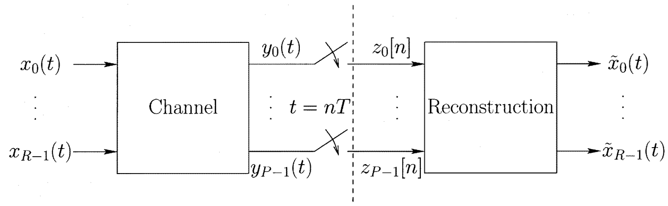

Fig. 1. MIMO sampling problem.

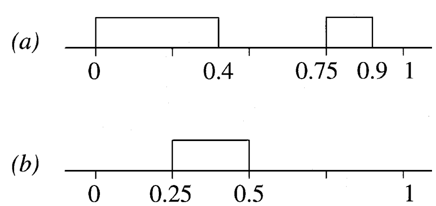

Fig. 2. Example multiband spectra $X_0(f)$ and $X_1(f)$ of the inputs to a two-input MIMO channel.

assumed to be continuous-time signals. The channel is modeled as a linear time-invariant system. The channel outputs are sampled at a uniform rate of $1/T$ to produce discrete-time sequences $z_p[n]$. From a practical viewpoint, we can interpret this as the sampling step prior to processing digitally. The reconstruction block, which is shown to the right of the dashed line, inverts the MIMO channel to produce estimates $\tilde{x}_r(t)$ of the input signals. The MIMO channel and the specral supports of the inputs are assumed to be given and fixed by the nature of the problem. We have the freedom to choose the sampling scheme for the outputs of the channel and design the reconstruction system. We emphasize that the focus of this paper is on sampling, motivated by the interest in all-digital processing. Hence, we assume that continuous-time operations such as filtering and modulation of the continuous-time MIMO outputs (or the inputs) prior to sampling them are not allowed.

As appropriate in many applications, we assume the input signals are multiband, with possibly different band structure (spectral support) for the different inputs. Fig. 2 shows such an example for a two-input MIMO channel, which will be used throughout the paper for illustrative purposes. (In this example, the spectra are one-sided, i.e., supported only on positive frequencies, so that the signals are complex-valued.)

The problem we address in this paper is a special, uniform sampling case of the general MIMO sampling problem introduced in [29] and [30]. We study the following issues in this paper: a) the relation of stable MIMO sampling to frame theory and b) the necessary and sufficient conditions on the channel allowing to achieve perfect reconstruction of the inputs under uniform sampling. Even though we consider only *uniform* sampling of the MIMO channel outputs, we will see later that this sampling scheme is fairly general, and it encompasses most periodic nonuniform sampling of the channel outputs, with sampling at different rates on different channels.

We derived necessary sampling density conditions for the general MIMO sampling problem in [29] and [30]. We showed that stable sampling and reconstruction of the inputs imposes lower bounds on the sampling densities on the various channels, regardless of whether the sampling is uniform or not. These results are analogues of Landau's classic minimum density results for multiband single-channel sampling [26]. It is not clear whether those conditions are sufficient; however, they indicate the potential for reduction in the sampling density needed for stable sampling, relative to the Nyquist rate sampling of each channel output. In this paper, we demonstrate how to achieve stable sampling and reconstruction at rates close to the minimum density. We can think of these results as partial sufficient conditions for stable MIMO sampling, although we do not provide explicit bounds on the sampling densities themselves. These results thus complement our results in [29].

This paper is organized as follows. Section II formulates the problems and introduces some notation and definitions used in the rest of the paper. In Section III, we present models for the channel and reconstruction, demonstrating that various nonuniform sampling schemes can be reduced to uniform sampling of the outputs of a modified channel. Section IV deals with the problem of perfect reconstruction of the channel inputs. We explore the connection between MIMO sampling and frame theory. The computation of the frame bounds enables us to determine necessary conditions on the input signal spaces, the channel characteristics, and the sampling rate for the existence of reconstruction filters that achieves stable and perfect reconstruction of the inputs. We also present additional conditions under which there exist reconstruction filters that are continuous in the frequency domain. This, as elaborated upon later, is important from the viewpoint of finite impulse response (FIR) filter design.

II. Definitions and Notation

We denote the Fourier transform of a continuous-time square-integrable signal $x(t)$ by

$$X(f) = \int_{\mathbb{R}} x(t) e^{-j2\pi f t} dt.$$

Similarly, for a discrete-time signal $y[n]$, we define its Fourier transform to be

$$Y[\nu] = \sum_{n \in \mathbb{Z}} y[n] e^{-j2\pi \nu n}.$$

In general, we denote discrete-time and continuous-time signals (either scalar-valued or vector-valued) using lower-case letters and their Fourier transforms by the corresponding upper-case letters. Let the class of continuous, finite-energy signals bandlimited to the set of frequencies $\mathcal{F}$ be

$$\mathcal{B}(\mathcal{F}) = \{x \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : X(f) = 0, \forall f \notin \mathcal{F}\}. \quad (1)$$

Denote the class of complex-valued matrices of size $M \times N$ by $\mathbb{C}^{M \times N}$, the conjugate-transpose of $\boldsymbol{A}$ by $\boldsymbol{A}^H$, and its pseudo inverse by $\boldsymbol{A}^{\dagger}$. The identity matrix of size $N \times N$ is denoted by $\boldsymbol{I}_N$ and the zero matrix by $\mathbf{0}$. Let $\lambda_{\max}(\cdot)$ and $\lambda \min(\cdot)$ denote the largest and smallest eigenvalues of a matrix with real eigenvalues.

For a given matrix $\boldsymbol{A}$, let $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$ denote the submatrix of $\boldsymbol{A}$ corresponding to rows indexed by the set $\mathcal{R}$ and columns by the set $\mathcal{C}$. The quantity $\boldsymbol{A}_{\bullet,\mathcal{C}}$ denotes a submatrix formed by keeping all rows of $\boldsymbol{A}$ but only columns indexed by $\mathcal{C}$, whereas $\boldsymbol{A}_{\mathcal{R},\bullet}$

denotes the submatrix formed by retaining rows indexed by $\mathcal{R}$ and all columns. We use a similar notation for vectors. Hence, $\boldsymbol{X}_{\mathcal{R}}$ is the subvector of $\boldsymbol{X}$ corresponding to rows indexed by $\mathcal{R}$. We always apply the subscripts of a matrix before the superscript. Therefore, $\boldsymbol{A}^H_{\mathcal{R},\mathcal{C}}$ is the conjugate-transpose of $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$. When dealing with singleton index sets $\mathcal{R} = \{r\}$ or $\mathcal{C} = \{c\}$, we omit the curly braces for readability. Therefore, $\boldsymbol{A}_{r,\bullet}$ and $\boldsymbol{A}_{\bullet,c}$ are the rth row and the cth column of $\boldsymbol{A}$, respectively. For convenience, we always number the rows and columns of a finite-size matrix starting from 0. For infinite-size matrices, the row and column indices range over $\mathbb{Z}$.

We denote the indicator function by $\chi(\cdot)$. Next, suppose that $\mathcal{S}$ is a subset of $\mathbb{R}$ or $\mathbb{Z}$, and a is an element of $\mathbb{R}$ or $\mathbb{Z}$. Then

$$\begin{aligned}\mathcal{S} \oplus a &= \{s + a : s \in \mathcal{S}\}\\ \mathcal{S} \ominus a &= \{s - a : s \in \mathcal{S}\}\\ a\mathcal{S} &= \{as : s \in \mathcal{S}\}\\ \mathcal{S} \bmod a &= \{s \bmod a : s \in \mathcal{S}\}.\end{aligned}$$

denote, respectively, the positive and negative translations, scaling, and the modulus of $\mathcal{S}$ by a. Let $\mu(\mathcal{S})$ denote the Lebesgue measure and int $\mathcal{S}$ and $\overline{\mathcal{S}}$ the interior and closure of a set $\mathcal{S} \subseteq \mathbb{R}$, respectively. Let $|\mathcal{S}|$ denote the cardinality of a finite set $\mathcal{S}$. Finally, let ess inf and ess sup denote the essential infimum and supremum, i.e.,

$$\begin{aligned}\text{ess inf } g(t) &= \sup\{\gamma : g(t) \geq \gamma \text{ a.e.}\}\\ \text{ess sup } g(t) &= \inf\{\gamma : g(t) \leq \gamma \text{ a.e.}\}\end{aligned}$$

for any real function g, where "a.e." stands for *almost everywhere*.

III. Sampling and Reconstruction Models

Let the input and output signals of the MIMO channel depicted in Fig. 1 be represented in vector form as

$$\begin{aligned}\boldsymbol{x}(t) &= \Big(x_0(t)\; x_1(t)\; \cdots\; x_{R-1}(t)\Big)^T\\ \boldsymbol{y}(t) &= \Big(y_0(t)\; y_1(t)\; \cdots\; y_{P-1}(t)\Big)^T.\end{aligned} \tag{2}$$

For convenience, define $\mathcal{R} = \{0, 1, \ldots, R-1\}$ and $\mathcal{P} = \{0, 1, \ldots, P-1\}$. These sets index the components of the input and the output vectors. For each $r \in \mathcal{R}$, we model $x_r(t)$ as a *multiband signal* $x_r(t) \in \mathcal{B}(\mathcal{F}_r)$, where the spectral support $\mathcal{F}_r$ is a finite union of disjoint intervals:

$$\mathcal{F}_r = \bigcup_{n=1}^{N_r} [a_{rn}, b_{rn}), \quad a_{r1} < b_{r1} < a_{r2} < \cdots < a_{rN_r} < b_{rN_r}. \tag{3}$$

We model the MIMO channel as a linear and shift-invariant system. Thus, we can write

$$\boldsymbol{y}(t) = \boldsymbol{g} * \boldsymbol{x}(t) = \int_{\mathbb{R}} \boldsymbol{g}(t-\tau)\boldsymbol{x}(\tau)d\tau$$

where $*$ denotes convolution, and

$$\boldsymbol{g}(t) = \begin{pmatrix} g_{0,0}(t) & \cdots & g_{0,R-1}(t)\\ \vdots & \ddots & \vdots\\ g_{P-1,0}(t) & \cdots & g_{P-1,R-1}(t)\end{pmatrix} \in \mathbb{C}^{P\times R}$$

is the impulse response matrix of the channel. Therefore

$$\boldsymbol{Y}(f) = \boldsymbol{G}(f)\boldsymbol{X}(f), \tag{4}$$

where $\boldsymbol{X}(f), \boldsymbol{Y}(f)$, and $\boldsymbol{G}(f)$ are the Fourier transforms of $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$, and $\boldsymbol{g}(t)$, respectively. In particular, $\boldsymbol{G}(f)$ is the *channel transfer function matrix*. The channel outputs are sampled at $t = nT$, $n \in \mathbb{Z}$, and we denote these output quantities by $z_p[n] = y_p(nT)$ or in matrix form by

$$\boldsymbol{z}[n] \stackrel{\text{def}}{=} \Big(z_0[n]\; z_1[n]\; \cdots\; z_{P-1}[n]\Big)^T = \boldsymbol{y}(nT), \quad n \in \mathbb{Z}.$$

Then, using (4), it is clear that

$$\begin{aligned}\boldsymbol{Z}[\nu] &= \frac{1}{T}\sum_{l\in\mathbb{Z}} \boldsymbol{Y}\Big(\frac{\nu + l}{T}\Big), \quad \nu \in [0,1)\\ &= \frac{1}{T}\sum_{l\in\mathbb{Z}} \boldsymbol{G}\Big(\frac{\nu + l}{T}\Big)\boldsymbol{X}\Big(\frac{\nu + l}{T}\Big), \quad \nu \in [0,1).\end{aligned} \tag{5}$$

We model the reconstruction block as follows:

$$\tilde{\boldsymbol{x}}(t) = \sum_{n\in\mathbb{Z}} \boldsymbol{h}(t - nT)\boldsymbol{z}[n] \tag{6}$$

where

$$\boldsymbol{h}(t) = \begin{pmatrix} h_{0,0}(t) & \cdots & h_{0,P-1}(t)\\ \vdots & \ddots & \vdots\\ h_{R-1,0}(t) & \cdots & h_{R-1,P-1}(t)\end{pmatrix} \in \mathbb{C}^{R\times P}.$$

It is clear from (6) that the entire MIMO system (consisting of the channel, the samplers and the reconstruction block) is invariant to a time-shift by a multiple of T, i.e.,

$$\boldsymbol{x}(t) \rightarrow \tilde{\boldsymbol{x}}(t) \Rightarrow \boldsymbol{x}(t - nT) \rightarrow \tilde{\boldsymbol{x}}(t - nT), \quad \forall n \in \mathbb{Z}, t \in \mathbb{R}.$$

Conversely, (6) is the most general linear transformation that allows this invariance. Taking its Fourier transform and rewriting in matrix form yields

$$\tilde{\boldsymbol{X}}(f) = \boldsymbol{H}(f)\boldsymbol{Z}[fT], \quad f \in \mathbb{R} \tag{7}$$

where $\boldsymbol{H}(f)$, which is the Fourier transform of $\boldsymbol{h}(t)$, is the *reconstruction filter matrix*. Owing to the periodicity of $\boldsymbol{Z}[\nu]$, we can rewrite (7) as

$$\tilde{\boldsymbol{X}}\Big(f + \frac{l'}{T}\Big) = \boldsymbol{H}\Big(f + \frac{l'}{T}\Big)\boldsymbol{Z}[fT], \quad l' \in \mathbb{Z},\; f \in \Big[0, \frac{1}{T}\Big). \tag{8}$$

We can now rewrite (5) and (8) compactly as

$$\boldsymbol{Z}[fT] = \boldsymbol{\mathcal{G}}(f)\boldsymbol{\mathcal{X}}(f) \tag{9}$$

$$\tilde{\boldsymbol{\mathcal{X}}}(f) = \boldsymbol{\mathcal{H}}(f)\boldsymbol{Z}[fT] \tag{10}$$

for $f \in [0, 1/T)$, where $\boldsymbol{\mathcal{X}}(f)$ and $\tilde{\boldsymbol{\mathcal{X}}}(f)$ are the *modulated input and reconstructed vectors*, whose entries are

$$\boldsymbol{\mathcal{X}}_{Rl+r}(f) = X_r\left(f + \frac{l}{T}\right), \quad (r,l) \in \mathcal{R} \times \mathbb{Z} \tag{11}$$

$$\tilde{\boldsymbol{\mathcal{X}}}_{Rl+r}(f), = \tilde{X}_r\left(f + \frac{l}{T}\right), \quad (r,l) \in \mathcal{R} \times \mathbb{Z} \tag{12}$$

whereas $\boldsymbol{\mathcal{G}}(f)$ and $\boldsymbol{\mathcal{H}}(f)$ are the *modulated channel and reconstruction matrices*, whose entries are

$$\boldsymbol{\mathcal{G}}_{p,Rl+r}(f) = \frac{1}{T} G_{pr}\left(f + \frac{l}{T}\right), \; (p,r,l) \in \mathcal{P} \times \mathcal{R} \times \mathbb{Z} \tag{13}$$

$$\boldsymbol{\mathcal{H}}_{Rl+r,p}(f) = H_{rp}\left(f + \frac{l}{T}\right), \; (p,r,l) \in \mathcal{P} \times \mathcal{R} \times \mathbb{Z}. \tag{14}$$

Note that even though these matrices have infinitely many columns or rows, only a finite summation is involved in (9) because the components of $\boldsymbol{X}(f)$ are bandlimited, implying that only a finite number of entries in $\boldsymbol{\mathcal{X}}(f)$ are nonzero. In the next section, we seek conditions on the channel and the inputs that guarantee perfect reconstruction of the input signals or, equivalently, perfect inversion of the channel.

We consider only uniform sampling in this paper. Fortunately, most periodic nonuniform sampling schemes can be expressed as special cases of uniform sampling. To see this, consider the following situation where the pth channel output $y_p(t)$ is sampled at

$$t \in \Lambda_p = \{nT_p + \lambda_{kp} : k = 0, \ldots, K_p - 1\}.$$

The period of the sampling pattern for the pth output channel is T_p, and the average sampling density of the pth output is K_p/T_p. First, consider the case where all the periods are equal, i.e., $T_p = T$. Then, we can write

$$\Lambda_p = \bigcup_{k=0}^{K_p-1} (T\mathbb{Z} + \lambda_{kp}).$$

In other words, Λ_p is composed of a union of K_p uniform sampling sets of density $1/T$. Consider a hypothetical MIMO channel whose transfer function matrix is obtained by performing the following modification to $\boldsymbol{G}(f)$. We replace the pth row of $\boldsymbol{G}(f)$, namely $\boldsymbol{G}_{p,\bullet}(f)$, by the following K_p rows: $\boldsymbol{G}_{p,\bullet}(f)e^{-j2\pi f\lambda_{kp}}$, $k = 0, \ldots, K_p - 1$. The new channel matrix has $\sum_p K_p$ rows, and the samples of the new outputs taken at $t = nT$ are precisely equal to the samples of the old MIMO channel outputs taken on the periodic nonuniform sampling sets $\{\Lambda_p\}$ and reordered. Next, suppose that the different channels have unequal but *commensurate* sampling periods, i.e., that the ratios of sampling periods are rational numbers $T_p = (m_p/n_p)T$ for some $m_p, n_p \in \mathbb{N}$, and $T \in \mathbb{R}$. In this case, a common period for all the sampling sets $\{\Lambda_p\}$ is $T\prod n_p$, and an argument, as before, allows us to convert this to uniform sampling of the outputs of a hypothetical MIMO channel. The upshot of this argument is that most periodic nonuniform sampling (except those with noncommensurate periods) may be recast as a uniform sampling problem. Of course, the price to be paid is that the hypothetical MIMO channel has many more outputs. We illustrate this in the following example.

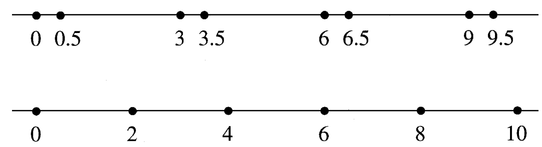

Fig. 3. Commensurate periodic nonuniform sampling sets.

Example 1: Let $\boldsymbol{G}(f)$ be the channel transfer function matrix of a MIMO channel with $P = 2$ outputs. Let the sampling sets for the channel outputs be as depicted in Fig. 3, i.e.,

$$\Lambda_0 = \{3n, 3n + 0.5 : n \in \mathbb{Z}\}$$
$$\Lambda_1 = \{2n : n \in \mathbb{Z}\}.$$

These sets are clearly commensurate because sampling periods $T_0 = 3$ and $T_1 = 2$ are such that T_0/T_1 is rational. A common period for the two sampling sets is obviously 6. Indeed, we have

$$\Lambda_0 = \bigcup_{k=0}^{3} (6\mathbb{Z} + \lambda_{k,0}), \; \{\lambda_{k,0} : k = 0, \ldots, 3\} = \{0, 0.5, 3, 3.5\}$$

$$\Lambda_1 = \bigcup_{k=0}^{2} (6\mathbb{Z} + \lambda_{k,1}), \; \{\lambda_{k,1} : k = 0, \ldots, 2\} = \{0, 2, 4\}.$$

Hence, the modified channel has six outputs, and the rows of its transfer function matrix $\tilde{\boldsymbol{G}}(f)$ are given by

$$\tilde{\boldsymbol{G}}_{k,\bullet}(f) = \boldsymbol{G}_{0,\bullet}(f)e^{-j2\pi f\lambda_{k,0}}, \quad k = 0, 1, 2, 3$$
$$\tilde{\boldsymbol{G}}_{k+4,\bullet}(f) = \boldsymbol{G}_{1,\bullet}(f)e^{-j2\pi f\lambda_{k,1}}, \quad k = 0, 1, 2.$$

If the outputs of the hypothetical channel are sampled uniformly at $t = 6n$, $n \in \mathbb{Z}$, we essentially obtain a reordered sequence of the samples of the original MIMO channel outputs taken on the samples sets Λ_1 and Λ_2.

We have shown that commensurate periodic nonuniform sampling is really uniform sampling in disguise because their equivalence is shown using the above modification trick. Therefore, the study of uniform sampling automatically provides answers to the commensurate periodic nonuniform sampling problem. In the subsequent sections, we present results for uniform MIMO sampling only.

In practice, we would usually only attempt to reconstruct a version of the set of inputs that is uniformly sampled at a sufficiently high rate and implement $\boldsymbol{H}(f)$ using FIR filters. The continuous-time version could then be reconstructed by a bank of conventional D/A converters on the reconstructed discrete-time signals. In particular, it would be desirable to use a reconstruction filter matrix $\boldsymbol{H}(f)$ that is continuous in f. The reason for this is roughly the following. Recall that a real function on a real compact set can be approximated arbitrarily closely (in the L^∞ sense) by polynomials if the given function is continuous. Similarly, if $\boldsymbol{H}(f)$ is continuous in f, we can approximate the matrix function $\boldsymbol{H}(f)$ arbitrarily closely in the $\mathcal{H}^\infty$ sense (and thus ensure an arbitrarily small worst-case $\mathcal{L}^2$ reconstruction error) by choosing sufficiently long FIR filters. Although we will not delve into implementation issues in this paper, we do consider both cases (*with* and *without* the continuity requirement imposed on $\boldsymbol{H}(f)$) in the next section, when we derive conditions for perfect reconstruction.

IV. Perfect Reconstruction

A. Preliminaries

We begin with some definitions. Define the following two *spectral index sets* at frequency $f \in [0, 1/T)$:

$$\mathcal{K}_f^\circ = \left\{ (r,l) \in \mathcal{R} \times \mathbb{Z} : \left(f + \frac{l}{T} \right) \in \mathcal{F}_r \right\}$$
$$\mathcal{K}_f = \left\{ Rl + r : (r,l) \in \mathcal{K}_f^\circ \right\}. \tag{15}$$

In other words, $\mathcal{K}_f$ is a function of f whose entries are indices l of the nonvanishing elements in the list $\{X(F + l/T)\}$. Note that $\mathcal{K}_f^\circ$ and $\mathcal{K}_f$ contain the same information because the map from the pair $(r,l) \in \mathcal{R} \times \mathbb{Z}$ to a single index $Rl + r \in \mathbb{Z}$ is invertible. In addition, let $\mathcal{K}_f^c = \mathbb{Z} \backslash \mathcal{K}_f$ denote the complement of $\mathcal{K}_f$. We now have the following proposition.

Proposition 1: Suppose that sets $\mathcal{F}_r$, $r \in \mathcal{R}$ have multiband structure, as defined in (3), and that T is the sampling interval size. Then, $\mathcal{K}_f$ is piecewise constant on [0, $1/T$), i.e., there exists a collection of disjoint intervals $\mathcal{I}_m$ of the form $[\alpha, \beta)$, and sets $\mathcal{K}^m$, $m = 1, \ldots, M$ such that $\mathcal{K}_f = \mathcal{K}^m$ for $f \in \mathcal{I}_m$ and

$$\bigcup_{m=1}^{M} \mathcal{I}_m = \left[0, \frac{1}{T}\right).$$

This result is easily demonstrated by using an argument very similar to the one in [25] for multicoset sampling. Hence, we can write

$$\mathcal{I}_m = [\gamma_m, \gamma_{m+1}), \quad m \in \mathcal{M}$$
$$\gamma_1 < \gamma_2 < \cdots < \gamma_{M+1}$$

such that $\gamma_1 = 0$, and $\gamma_{M+1} = 1/T$.

Example 2: In this example, we illustrate the sets in (15) and the modulated input vectors for a simple case. Consider a MIMO channel with $R = 2$ inputs, and input spectra $X_0(f)$ and $X_1(f)$ that have supports as illustrated in Fig. 2, i.e.,

$$\mathcal{F}_0 = [0, 0.4) \cup [0.75, 0.9) \quad \text{and} \quad \mathcal{F}_1 = [0.25, 0.5).$$

Let the sampling period be $T = 4$. For this choice, it is easy to verify that

$$\mathcal{I}_1 = [0, 0.15) \quad \text{and} \quad \mathcal{I}_2 = [0.15, 0.25).$$

Furthermore, (15) and Proposition 1 imply that

$$\mathcal{K}_f^\circ = \begin{cases} \{(0,0),(0,1),(0,3),(1,1)\}, & \text{if } f \in \mathcal{I}_1 \\ \{(0,0),(1,1)\}, & \text{if } f \in \mathcal{I}_2. \end{cases}$$

Therefore, $\mathcal{K}_1 = \{0,2,6,3\}$, and $\mathcal{K}_2 = \{0,3\}$. Finally, we illustrate the vector $\boldsymbol{\mathcal{X}}(f)$

$$\boldsymbol{\mathcal{X}}(f) = \begin{pmatrix} \vdots \\ \mathcal{X}_{-2}(f) \\ \mathcal{X}_{-1}(f) \\ \mathcal{X}_0(f) \\ \mathcal{X}_1(f) \\ \mathcal{X}_2(f) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ X_0\left(f - \frac{1}{4}\right) \\ X_1\left(f - \frac{1}{4}\right) \\ X_0(f) \\ X_1(f) \\ X_0\left(f + \frac{1}{4}\right) \\ \vdots \end{pmatrix}.$$

For $f \in \mathcal{I}_1$, the only nonvanishing components of $\boldsymbol{\mathcal{X}}(f)$ are $\mathcal{X}_0(f)$, $\mathcal{X}_2(f)$, $\mathcal{X}_3(f)$, and $\mathcal{X}_0(f)$, whereas for $f \in \mathcal{I}_2$, they are $\mathcal{X}_0(f)$ and $\mathcal{X}_3(f)$.

In the sequel, we derive the conditions on the channel and the spectral supports $\mathcal{F}_r$ of the channel inputs for the existence of a reconstruction filter matrix $\boldsymbol{H}(f)$ that achieves perfect reconstruction of the inputs. We consider both cases *with* and *without* the continuity requirement imposed on the channel and reconstruction filters. As we will see later, the continuity of $\boldsymbol{H}(f)$ may also require the continuity of the channel transfer function matrix $\boldsymbol{G}(f)$.

Necessary Condition for Perfect Reconstruction: In the next subsection, we use frame theory to derive necessary and sufficient conditions for stable and perfect reconstruction of the channel inputs, but we first present a simple necessary condition. From (15), it is clear that all the nonzero entries of $\boldsymbol{\mathcal{X}}(f)$ are captured in the subvector $\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)$, and hence, we can rewrite (9) and (10) together as

$$\tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{K}_f}(f) = \boldsymbol{\mathcal{H}}_{\mathcal{K}_f,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f) \tag{16}$$
$$\text{and} \quad \tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{K}_f^c}(f) = \boldsymbol{\mathcal{H}}_{\mathcal{K}_f^c,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f). \tag{17}$$

For perfect reconstruction, we require the existence of $\boldsymbol{\mathcal{H}}(f)$ such that $\tilde{\boldsymbol{\mathcal{X}}}(f) = \boldsymbol{\mathcal{X}}(f)$ a.e. It is clear that this would happen if and only if

$$\boldsymbol{\mathcal{H}}_{\mathcal{K}_f,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) = \boldsymbol{I}_{|\mathcal{K}_f|} \quad \text{a.e.}$$
$$\boldsymbol{\mathcal{H}}_{\mathcal{K}_f^c,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) = \boldsymbol{0} \quad \text{a.e.} \tag{18}$$

where $|\mathcal{K}_f|$ is the number of elements of $\mathcal{K}_f$. This can be expressed more compactly as

$$\boldsymbol{\mathcal{H}}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) = \boldsymbol{I}_{\bullet,|\mathcal{K}_f|}. \tag{19}$$

Since $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) \in \mathbb{C}^{P \times |\mathcal{K}_f|}$, we require that $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)$ have full column rank a.e. This condition guarantees that a solution (possibly nonunique) to (19) exists. In view of Proposition 1, we now obtain the following necessary condition for perfect reconstruction:

$$\text{rank}\left(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)\right) = |\mathcal{K}^m|, \quad \text{a.e. } f \in \mathcal{I}_m. \tag{20}$$

However, this condition does not address the issue of stability of reconstruction and, hence, may be insufficient.

Example 3: The necessary conditions reduce to a familiar form for the special case of a single-input, single-output (SISO) channel, with $R = P = 1$. This case then corresponds to (single channel) deconvolution of a multiband signal $x \in \mathcal{B}(\mathcal{F})$ from the sampled output y. The Fourier transforms $X(f)$ and $Y(f)$ of the channel input and the output, respectively, and the channel transfer function $G(f)$ are all scalar in this case. Thus, the spectral index set defined in (15) reduces to $\mathcal{K}_f = \left\{ l : f + l/T \in \mathcal{F} \right\}$, and the modulated channel matrix $\boldsymbol{\mathcal{G}}(f)$ has only one row. Hence, the necessary condition for perfect reconstruction in (20) is equivalent to the following set of conditions:

$$|\mathcal{K}_f| \le 1 \quad \text{and} \quad G(f) \ne 0, \quad f \in \mathcal{F}. \tag{21}$$

The first condition says that there must be no aliasing of $\mathcal{F}$ due to sampling, and the second one says that the channel transfer function cannot have any nulls on the set $\mathcal{F}$.

These conditions can be easily rederived "from first principles" as follows. Suppose that any $x \in \mathcal{B}(\mathcal{F})$ can be reconstructed from the samples of y. Then, x can also be reconstructed from y itself, but this is only possible if (4) can be inverted. The necessary condition on $G(f)$ then follows. Returning to the assumption that x can be reconstructed from the samples of y, it follows that y can also be obtained using (4). However, as we know from classical results on uniform multiband sampling, y can only be obtained from its uniform samples if its spectrum is not aliased. Noting that, owing to the condition on $G(f)$, $Y(f)$ is supported on $\mathcal{F}$, the nonaliasing condition in (21) follows.

These conditions do not, however, guarantee stability of inversion. For instance, if $G(f)$ takes arbitrarily small or large values for $f \in \mathcal{F}$, we cannot invert (4) in a stable way. We study the stability and present necessary and sufficient conditions immediately following this example.

Finally, specializing the example further to pure single channel sampling, consider the case $G(f) = 1$. In this case, the necessary condition on $G(f)$ holds trivially. The only other necessary condition is the no-aliasing condition $|\mathcal{K}_f| \leq 1$ for perfect inversion. Incidentally, this condition is both necessary and sufficient for stable inversion, as we know for the classical problem of uniform multiband sampling.

B. Stable Sampling

The MIMO channel can be viewed as a linear transformation (operator) from the class of input signals to the space of its samples. The condition in (20) on the channel and the input signals is necessary for stable perfect reconstruction. However, it does not suffice because it does not answer the important question regarding stability of the reconstruction. In this section, we will use frame theory to study the stability of the MIMO sampling problem. Recall the definition of a frame.

Definition 1: Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{\psi_n\} \subseteq \mathbb{H}$ is a frame if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_n |\langle \psi_n, x\rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathbb{H}$. If $A = B$, then the frame is a tight frame.

The frame operator S, which is defined as

$$Sx = \sum_n \langle x, \psi_n\rangle \psi_n, \quad \forall x \in \mathbb{H}$$

is a bounded linear operator satisfying $AI \leq S \leq BI$, where I is the identity operator. Define $\tilde{\psi}_n = S^{-1}\psi_n$. Then, $\{\tilde{\psi}_n\}$ is also a frame (the *dual frame*) for $\mathbb{H}$ with frame bounds B^{-1} and A^{-1}, and any $x \in \mathbb{H}$ can be expressed as

$$x = \sum_n \langle x, \tilde{\psi}_n\rangle \psi_n = \sum_n \langle x, \psi_n\rangle \tilde{\psi}_n. \tag{22}$$

In the context of MIMO sampling, the relevant Hilbert space is the class of input signals:

$$\mathbb{H} = \mathcal{B}(\mathcal{F}_0) \times \cdots \times \mathcal{B}(\mathcal{F}_{R-1}).$$

The inner product and norm on $\mathbb{H}$ are defined as

$$\langle \boldsymbol{x}, \boldsymbol{w}\rangle = \int_{\mathbb{R}} \boldsymbol{w}^H(t)\boldsymbol{x}(t)dt, \quad \boldsymbol{x}, \boldsymbol{w} \in \mathbb{H}$$
$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}.$$

We now present an important definition for the stability of MIMO sampling (see [29]).

Definition 2: The MIMO sampling scheme is called *stable* if there exist constants $A, B > 0$ such that

$$A\|\boldsymbol{x}\|^2 \leq \sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2 \leq B\|\boldsymbol{x}\|^2 \tag{23}$$

for all $\boldsymbol{x}(t) \in \mathbb{H}$.

This definition is readily verified to correspond to the requirement that the linear operator mapping signals in $\mathbb{H}$ to the samples of the channel output be a bounded linear operator and have a bounded inverse. Suppose that A and B are the smallest and largest such constants, i.e., the best frame bounds. Then, the ratio $K = \sqrt{B/A} \geq 1$ is the *condition number* of this linear operator, and we call it the condition number of the MIMO sampling scheme. It depends on the properties of both the channel and the sampling of its outputs. As is well known in linear algebra [31], [32], K^2 is a bound on the amplification of the normalized error energy due to the reconstruction filters. It follows that stability of reconstruction as defined above implies that the condition number is finite and that errors in the inputs or in the sampled outputs cannot produce arbitrarily large errors in the reconstructed inputs.

Next, we consider the frame-theoretic implications of (23) stable MIMO reconstruction. Define the diagonal matrix

$$\boldsymbol{J}(f) = \mathrm{diag}\big(\chi(f \in \mathcal{F}_0), \ldots, \chi(f \in \mathcal{F}_{R-1})\big) \tag{24}$$

where $\chi(f \in \mathcal{F}_r)$ is the indicator function of the set $\mathcal{F}_r$. Then, we have

$$\boldsymbol{J}(f)\boldsymbol{X}(f) = \boldsymbol{X}(f) \tag{25}$$

because $X_r(f)$ is supported on $\mathcal{F}_r$. In view of (25), we can rewrite $z_p[n]$ as

$$\begin{aligned} z_p[n] = y_p(nT) &= \int_{\mathbb{R}} e^{j2\pi fnT}\boldsymbol{G}_{p,\bullet}(f)\boldsymbol{X}(f)df \\ &= \int_{\mathbb{R}} e^{j2\pi fnT}\boldsymbol{G}_{p,\bullet}(f)\boldsymbol{J}(f)\boldsymbol{X}(f)df \\ &= \int_{\mathbb{R}} \boldsymbol{\Psi}_{pn}^H(f)\boldsymbol{X}(f)df \end{aligned} \tag{26}$$

where

$$\begin{aligned} &\boldsymbol{\Psi}_{pn}^H(f) = e^{j2\pi fnT}\boldsymbol{G}_{p,\bullet}(f)\boldsymbol{J}(f) \\ \iff\ &\boldsymbol{\psi}_{pn}(t) = \int_{\mathbb{R}} \boldsymbol{J}(f)\boldsymbol{G}_{p,\bullet}^H(f)e^{j2\pi f(t-nT)}df \end{aligned} \tag{27}$$

for $(p, n) \in \mathcal{P} \times \mathbb{Z}$. It is clear that $\boldsymbol{\psi}_{pn} \in \mathbb{H}$. Using Parseval's theorem and (27), we conclude that

$$\langle \boldsymbol{x}, \boldsymbol{\psi}_{pn}\rangle = \int_{\mathbb{R}} \boldsymbol{\psi}_{pn}^H(t)\boldsymbol{x}(t)dt = \int_{\mathbb{R}} \boldsymbol{\Psi}_{pn}^H(f)\boldsymbol{X}(f)df = z_p[n].$$

Thus, $z_p[n]$ can be expressed as an inner product of $\boldsymbol{x}$ and $\boldsymbol{\psi}_{pn} \in \mathbb{H}$, and consequently, (23) is equivalent to the condition that $\{\boldsymbol{\psi}_{pn}\}$ forms a frame for $\mathbb{H}$. Suppose we denote its dual frame by $\{\tilde{\boldsymbol{\psi}}_{np} : n \in \mathbb{Z},\ p \in \mathcal{P}\}$; then, (22) yields the following reconstruction formula:

$$\boldsymbol{x} = \sum_{n\in\mathbb{Z}}\sum_{p\in\mathcal{P}} \langle \boldsymbol{x}, \boldsymbol{\psi}_{np}\rangle \tilde{\boldsymbol{\psi}}_{np} = \sum_{n\in\mathbb{Z}}\sum_{p\in\mathcal{P}} z_p[n]\tilde{\boldsymbol{\psi}}_{np}.$$

Thus, the frame-theoretic approach provides a reconstruction formula and a bound on the output errors in terms of the condition number of the sampling operator.

C. Conditions for Perfect Reconstruction

Our next result provides necessary and sufficient conditions on the channel matrix for stable MIMO reconstruction. Since our analysis will rely on the modulated channel and reconstruction matrices $\boldsymbol{\mathcal{G}}(f)$ and $\boldsymbol{\mathcal{H}}(f)$, the following proposition will turn out to be useful.

Proposition 2: If $G_{\bullet,r}(f)$ is continuous on $\overline{\mathcal{F}}_r$, then $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ is continuous on $\overline{\mathcal{I}}_m$, and the following "boundary condition" holds:

$$\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}\left(\frac{1}{T}\right) = \boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}\oplus R}(0),\ \mathcal{K} \subseteq \mathbb{Z}. \tag{28}$$

The quantity $\boldsymbol{H}(f)$ is continuous if and only if the entries of $\boldsymbol{\mathcal{H}}(f)$ are continuous on [0, $1/T$] and satisfy the boundary condition

$$\boldsymbol{\mathcal{H}}_{\mathcal{K},\bullet}\left(\frac{1}{T}\right) = \boldsymbol{\mathcal{H}}_{\mathcal{K}\oplus R,\bullet}(0),\ \mathcal{K} \subseteq \mathbb{Z}. \tag{29}$$

We do not care about $G_{pr}(f)$ outside the closure of the set $\mathcal{F}_r$ because $X_r(f)$ vanishes there. This explains why the conditions for $\boldsymbol{G}(f)$ and $\boldsymbol{H}(f)$ are different in Proposition 2. We omit its proof since it is quite straightforward, following directly from (13) and (14) and the definition of $\mathcal{K}^m$. The boundary conditions imply that the entries of the matrix $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}(0)$ are shifted versions of those of $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}(1/T)$, with a similar relationship for $\boldsymbol{\mathcal{H}}_{\mathcal{K},\bullet}$.

Theorem 1: The best frame bounds for the MIMO sampling problem are given by

$$A = T \operatorname*{ess\,inf}_{f\in[0,1/T]} \lambda_{\min}\left(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\right) \tag{30}$$

$$B = T \operatorname*{ess\,sup}_{f\in[0,1/T]} \lambda_{\max}\left(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\right). \tag{31}$$

In particular, $A > 0$ and $B < \infty$ are necessary and sufficient conditions for stable reconstruction of the MIMO inputs.

Proof: We need to compute

$$A = \inf_{\boldsymbol{x}\in\mathcal{B}} \sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2 \text{ and } B = \sup_{\boldsymbol{x}\in\mathcal{B}} \sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2 \tag{32}$$

where $\mathcal{B}$ is the set of input signals of unit combined energy:

$$\mathcal{B} = \left\{\boldsymbol{x} \in \mathbb{H} : \|\boldsymbol{x}\| = 1\right\}. \tag{33}$$

First, observe that

$$\begin{aligned}\|\boldsymbol{x}\|^2 &= \int_{\mathbb{R}} \|\boldsymbol{x}(t)\|^2 dt = \int_{\mathbb{R}} \|\boldsymbol{X}(f)\|^2 df \\ &\overset{(a)}{=} \int_{[0,1/T]} \|\boldsymbol{\mathcal{X}}(f)\|^2 df \\ &\overset{(b)}{=} \int_{[0,1/T]} \|\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df\end{aligned} \tag{34}$$

where the norms on the right-hand side of (34) are the Euclidean norms. The equality (a) above follows from (11), and (b) follows because $\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)$ captures all the nonzero entries of $\boldsymbol{\mathcal{X}}(f)$. Next

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2 &= \int_{\nu\in[0,1]} \|\boldsymbol{z}[\nu]\|^2 d\nu \\ &\overset{(a)}{=} T\int_{[0,1/T]} \|\boldsymbol{z}[TV]\|^2 df \\ &\overset{(b)}{=} T\int_{[0,1/T]} \|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df\end{aligned} \tag{35}$$

where (a) is obtained by a change of variables, and (b) from (9) and the fact that $\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)$ captures all the nonzero entries of $\boldsymbol{\mathcal{X}}(f)$. Therefore, (32)–(35) yield

$$\begin{aligned}A &= \inf T\int_{[0,1/T]} \|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df \\ &\quad \text{s.t.} \int_{[0,1/T]} \|\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df = 1 \\ B &= \sup T\int_{[0,1/T]} \|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df \\ &\quad \text{s.t.} \int_{[0,1/T]} \|\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df = 1.\end{aligned}$$

Now, the claimed results in (30) and (31) follow immediately. ■

Note that a simple necessary condition for perfect reconstruction is that $P \geq |\mathcal{K}^m|$ for each $m \in \mathcal{M}$. Clearly, multiple solutions $\boldsymbol{\mathcal{H}}(f)$ exist to (19) if $P > |\mathcal{K}^m|$ for some m. The average sampling density for this sampling scheme is P/T. Now, (15) implies that

$$|\mathcal{K}_f| = \sum_{r\in\mathcal{R}}\sum_{l\in\mathbb{Z}} \chi\left(f + \frac{l}{T} \in \mathcal{F}_r\right).$$

Hence

$$\int_{[0,1/T]} |\mathcal{K}_f| df = \sum_{r=0}^{R-1} \int_R \chi(f \in \mathcal{F}_r) = \sum_{r=0}^{R-1} \mu(\mathcal{F}_r) \tag{36}$$

where $\mu(\cdot)$ denotes the Lebesgue measure. Suppose that $P = |\mathcal{K}^m|$ for all m; then, (36) reduces to

$$\frac{P}{T} = \sum_{r=0}^{R-1} \mu(\mathcal{F}_r)$$

which coincides with the minimum necessary density for stable MIMO sampling [29] using any sampling scheme for the channel outputs, whether uniform or not. In addition, note that we have uniqueness of the reconstruction filters when $P = |\mathcal{K}^m|$.

The following corollary to Theorem 1 provides a simpler sufficient condition for the stability of the MIMO sampling scheme.

Corollary 1: Suppose that $\boldsymbol{G}(f)$ is such that $G_{pr}(f)$ is continuous for $f \in \overline{\mathcal{F}}_r$, and $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ has full column rank for all $m \in \mathcal{M}$, $f \in \overline{\mathcal{I}}_m = [\gamma_m, \gamma_{m+1}]$. Then, the MIMO sampling scheme is stable, i.e., $\{\boldsymbol{\psi}_{pn}\}$ forms a frame.

Proof: By Proposition 2, we have continuity of $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ on the compact set $\overline{\mathcal{I}}_m$. Therefore, both the smallest and the largest eigenvalues of $\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}^m}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ are continuous functions on $\overline{\mathcal{I}}_m$. Since the smallest eigenvalue is strictly positive for all $f \in [0, 1/T]$ by hypothesis, it follows that the infimum in (30) is attained, implying that $A > 0$. Similarly, $B < \infty$ because the supremum in (31) is attained. ■

We illustrate the MIMO sampling result of Theorem 1 for a simple MIMO channel.

Example 4: Consider a MIMO channel with $R = 2$ inputs and $P = 4$ outputs having the following transfer function matrix:

$$\boldsymbol{G}(f) = \begin{pmatrix} 1 & 1 \\ 1 & 1 + e^{-j2\pi f} \\ e^{-j2\pi f} & 0.25 + e^{-j4\pi f} \\ 1 + 0.5e^{-j2\pi f} & 1 + e^{-j4\pi f} \end{pmatrix}.$$

Let the input spectra $X_0(f)$ and $X_1(f)$ have supports as illustrated in Fig. 2, i.e.,

$$\mathcal{F}_0 = [0, 0.4) \cup [0.75, 0.9) \quad \text{and} \quad \mathcal{F}_1 = [0.25, 0.5).$$

Each output is a multiband signal supported on $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 = [0, 0.5) \cup [0.75, 0.9)$. Note that a naïve way to reconstruct the inputs is to first reconstruct the individual outputs and then invert the channel. This method requires a minimum sampling rate of $\mu(\mathcal{F}) = 0.65$ for each channel output. However, we demonstrate in this example that we do not need to sample each output at its minimum rate to achieve perfect reconstruction, but we can jointly reconstruct them from fewer samples. Let the sampling period be $T = 4$. For this choice, we have seen in Example 2 that $\mathcal{I}_1 = [0, 0.15)$, $\mathcal{I}_2 = [0.15, 0.25]$, and $\mathcal{K}_1 = \{0, 2, 6, 3\}$, $\mathcal{K}_2 = \{0, 3\}$. It can be verified numerically that

$$\begin{aligned} \operatorname{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_1}(f)\big) &= 4, \quad \forall f \in \overline{\mathcal{I}_1} \\ \operatorname{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_2}(f)\big) &= 2, \quad \forall f \in \overline{\mathcal{I}_2} \end{aligned}$$

where the matrices $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_1}(f)$ and $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_2}(f)$ are given by (37) and (38), shown at the bottom of the page.

Since $\boldsymbol{G}(f)$ is continuous, we conclude using Corollary 1 that stable perfect reconstruction of the inputs is possible from the channel output samples. Hence, it suffices to sample each output at a rate $1/T = 0.25$ for perfect stable reconstruction of the channel inputs, instead of the sampling them at a rate $\mu(\mathcal{F}) = 0.65$, which is required for the naïve approach. However, we see in Example 6 that the reconstruction filter matrix $\boldsymbol{H}(f)$ would necessarily have to be discontinuous. Finally, note that the total combined sampling density of the outputs is $P/T = 1$, whereas the minimum density, as dictated by [29], is $\mu(\mathcal{F}_0) + \mu(\mathcal{F}_1) = 0.8$.

In Example 4, we showed that the combined sampling density of 1 is achievable, but the lower bound on this density is 0.8. Therefore, we could potentially find a nonuniform MIMO sampling scheme that closes the gap. In fact, this is precisely what we are going to show in the following example.

Example 5: Let the inputs signal characteristics and the channel transfer function matrix be the same as in Example 4. In this specific example, we show that by using a proper nonuniform sampling strategy at the outputs, we can achieve the minimum combined sampling rate for all the output channels equal to the sum of measures of the input spectral supports [29]. Let the channel outputs be sampled on the sets $\Lambda_p = \{20n + \lambda_{kp} : k = 0, \ldots, K_p - 1\}$, where $(K_1, K_2, K_3, K_4) = (0, 3, 5, 8)$ and

$$\{\lambda_{kp} : 0 \le k < K_p\} = \begin{cases} \emptyset, & p = 0 \\ \{1, 8, 14\}, & p = 1 \\ \{2, 5, 8, 13, 18\}, & p = 2 \\ \{0, 2, 4, 5, 7, 8, 14, 17\}, & p = 3. \end{cases}$$

Evidently, these are all periodic nonuniform sampling sets having a common period of $T = 20$ and consisting of 16 cosets in all. Hence, the modified MIMO channel has a transfer function matrix $\tilde{\boldsymbol{G}}(f)$ of size 16×2, and its rows can be worked out as in Example 1. Since the band edges of $\mathcal{F}_0$ and $\mathcal{F}_1$ are all

$$\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_1}(f) = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 + e^{-j2\pi(f+1/4)} \\ e^{-j2\pi f} & e^{-j2\pi(f+1/4)} & e^{-j2\pi(f+3/4)} & 0.25 + e^{-j4\pi(f+1/4)} \\ 1 + 0.5e^{-j2\pi f} & 1 + 0.5e^{-j2\pi(f+1/4)} & 1 + 0.5e^{-j2\pi(f+3/4)} & 1 + e^{-j4\pi(f+1/4)} \end{pmatrix} \tag{37}$$

$$\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_2}(f) = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 + e^{-j2\pi(f+1/4)} \\ e^{-j2\pi f} & 0.25 + e^{-j4\pi(f+1/4)} \\ 1 + 0.5e^{-j2\pi f} & 1 + e^{-j4\pi(f+1/4)} \end{pmatrix}. \tag{38}$$

multiples of 0.05, we trivially obtain $M = 1$, $\mathcal{I}_1 = [0, 0.05)$, and

$$\mathcal{K}_1 = \{0,2,4,6,8,10,12,14,30,32,34\} \cup \{11,13,15,17,19\}.$$

Now, $\tilde{\boldsymbol{\mathcal{G}}}_{\bullet,\mathcal{K}_1}(f)$ is a continuous 16×16 matrix, whose rank is verifiable to be 16 for all f. By Corollary 1, we conclude that stable and perfect reconstruction of the channel inputs is possible from these periodic nonuniform MIMO samples. In fact, the stability bounds are $A = 8.0724 \times 10^{-4}$ and $B = 3.6833$, implying that the condition number $K = \sqrt{B/A} = 67.5487$. The sampling density of Λ_p is $d_p = K_p/T$ so that

$$(d_0, d_1, d_2, d_3) = (0, 0.15, 0.25, 0.5)$$

is an achievable point in density region for stable sampling. Obviously, the densities (d_0, d_1, d_2, d_3) must meet all the necessary conditions for stable sampling derived in [29]. In particular, the total combined sampling rate of all the outputs is $16/T = 0.8$, which is precisely equal to the minimum joint sampling density required, namely, $\mu(\mathcal{F}_0) + \mu(\mathcal{F}_1)$. Finally, we learn from this example that we need not sample the different outputs at the same rate. In fact, one of the channels is not sampled at all, unlike in Example 4, where, due to uniform sampling, we required samples from all channel outputs.

D. Existence of Continuous Solutions

Theorem 1 does not guarantee the existence of a continuous filter matrix $\boldsymbol{H}(f)$, which, as we have seen earlier, may be desirable from an implementation point of view. The following theorem shows that under a stronger set of conditions, we can guarantee the existence of a continuous filter matrix $\boldsymbol{H}(f)$. We begin with a lemma.

Lemma 1: Let $\boldsymbol{C}(f) \in \mathbb{C}^{p \times q}$ (with $p \geq q$) and $\boldsymbol{D}(f) \in \mathbb{C}^{r \times q}$ be matrix-valued continuous functions of $f \in [\alpha, \beta]$ such that rank $\boldsymbol{C}(f) = q$ for all $f \in [\alpha, \beta]$, and let $\boldsymbol{E}_\alpha, \boldsymbol{E}_\beta \in \mathbb{C}^{r \times p}$ be matrices satisfying

$$\boldsymbol{E}_\alpha \boldsymbol{C}(\alpha) = \boldsymbol{D}(\alpha) \quad \text{and} \quad \boldsymbol{E}_\beta \boldsymbol{C}(\beta) = \boldsymbol{D}(\beta).$$

Then, there exists a continuous $\boldsymbol{E}(f) \in \mathbb{C}^{r \times p}$ such that $\boldsymbol{E}(f)\boldsymbol{C}(f) = \boldsymbol{D}(f)$ for all $f \in [\alpha, \beta]$ and that $\boldsymbol{E}(\alpha) = \boldsymbol{E}_\alpha$ and $\boldsymbol{E}(\beta) = \boldsymbol{E}_\beta$ are satisfied.

The proof of Lemma 1 can be found in the Appendix. We can now derive the conditions for the existence of continuous reconstruction filters that achieve perfect reconstruction.

Theorem 2: Suppose that the MIMO transfer function matrix $\boldsymbol{G}(f)$ is such that $G_{pr}(f)$ is continuous for $f \in \overline{\mathcal{F}}_r$. Then, there exists a reconstruction filter matrix $\boldsymbol{H}(f)$ that is continuous in f that achieves stable and perfect reconstruction of the MIMO channel inputs if and only if

$$\text{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)\big) = |\mathcal{K}^m|, \ \forall f \in \text{int}\mathcal{I}_m = (\gamma_m, \gamma_{m+1}) \tag{39}$$
$$\text{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{J}_m}(\gamma_m)\big) = |\mathcal{J}_m|, \ m \in \mathcal{M}. \tag{40}$$

where

$$\begin{aligned} \mathcal{J}_m &= \mathcal{K}^m \cup \mathcal{K}^{m-1}, \ m = 2, \ldots, M \\ \mathcal{J}_1 &= \mathcal{K}_1 \cup (\mathcal{K}^M \oplus R). \end{aligned} \tag{41}$$

Proof: First, note that the hypotheses in this theorem are stronger than those of Corollary 1. Thus, stable reconstruction is guaranteed. We will first prove the necessity of (39) and (40). The first condition in (18) states that

$$\boldsymbol{\mathcal{H}}_{\mathcal{K}_f,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) = \boldsymbol{I}_{|\mathcal{K}_f|} \quad \text{a.e.} \tag{42}$$

Therefore, suppose that $\boldsymbol{H}(f)$ is a continuous solution of (42); then, Proposition 2 implies that $\boldsymbol{\mathcal{H}}_{\mathcal{K}^m,\bullet}(f)$ and $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ are continuous functions in the interior of $\mathcal{I}_m$, and, in fact, (42) must hold for all $f \in \text{int}\,\mathcal{I}_m$ and not just a.e., because both sides of (42) are continuous functions. Now, (39) follows immediately. Next, letting $f \downarrow \gamma_1 = 0$ in (18) and using the continuity of $\boldsymbol{\mathcal{H}}(f)$ gives us

$$\boldsymbol{\mathcal{H}}_{\mathcal{K}_1,\bullet}(0)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_1}(0) = \boldsymbol{I}_{|\mathcal{K}_1|}, \ \boldsymbol{\mathcal{H}}_{\mathcal{K}_1^c,\bullet}(0)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_1}(0) = \boldsymbol{0} \tag{43}$$

while letting $f \uparrow \gamma_{M+1} = 1/T$ in (18) instead, and using (28) and (29), we obtain

$$\begin{aligned} \boldsymbol{\mathcal{H}}_{\mathcal{K}^M \oplus R,\bullet}(0)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^M \oplus R}(0) &= \boldsymbol{I}_{|\mathcal{K}^M|} \\ \boldsymbol{\mathcal{H}}_{\mathcal{K}_M^c \oplus R,\bullet}(0)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^M \oplus R}(0) &= \boldsymbol{0}. \end{aligned} \tag{44}$$

Combining (43) and (44), we obtain the following set of necessary conditions:

$$\begin{aligned} \boldsymbol{\mathcal{H}}_{\mathcal{J}_1,\bullet}(0)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{J}_1}(0) &= \boldsymbol{I}_{|\mathcal{J}_M|} \\ \boldsymbol{\mathcal{H}}_{\mathcal{J}_1^c,\bullet}(0)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{J}_1}(0) &= \boldsymbol{0} \end{aligned} \tag{45}$$

where $\mathcal{J}_1 = \mathcal{K}_1 \cup (\mathcal{K}^M \oplus R)$. Using a similar continuity argument in the vicinity of γ_m for $m = 2, \ldots, M$ yields

$$\begin{aligned} \boldsymbol{\mathcal{H}}_{\mathcal{J}_m,\bullet}(\gamma_m)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{J}_m}(\gamma_m) &= \boldsymbol{I}_{|\mathcal{J}_m|} \\ \boldsymbol{\mathcal{H}}_{\mathcal{J}_m^c,\bullet}(\gamma_m)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{J}_m}(\gamma_m) &= \boldsymbol{0} \end{aligned} \tag{46}$$

where $\mathcal{J}_m = \mathcal{K}^m \cup \mathcal{K}^{m-1}$. Hence, (40) is necessary to meet conditions in (45) and (46).

To prove sufficiency of (39) and (40), we construct an appropriate reconstruction matrix $\boldsymbol{\mathcal{H}}(f)$ that is continuous in f and satisfies the boundary condition in (29), as well as the defining reconstruction conditions in (18). We first define the function $\boldsymbol{\mathcal{H}}(f)$ on the following finite set of frequencies $\{\gamma_m : m \in \mathcal{M}\}$:

$$\boldsymbol{\mathcal{H}}_{\mathcal{J}_m,\bullet}(\gamma_m) = \boldsymbol{\mathcal{G}}^\dagger_{\bullet,\mathcal{J}_m}(\gamma_m), \quad \boldsymbol{\mathcal{H}}_{\mathcal{J}_m^c,\bullet}(\gamma_m) = \boldsymbol{0}. \tag{47}$$

Then, to satisfy (29), we define

$$\begin{aligned} \boldsymbol{\mathcal{H}}_{\mathcal{J}_{M+1},\bullet}\left(\frac{1}{T}\right) &= \boldsymbol{\mathcal{H}}_{\mathcal{J}_1,\bullet}(0) = \boldsymbol{\mathcal{G}}^\dagger_{\bullet,\mathcal{J}_1}(0) \\ \boldsymbol{\mathcal{H}}_{\mathcal{J}_{M+1}^c,\bullet}\left(\frac{1}{T}\right) &= \boldsymbol{\mathcal{H}}_{\mathcal{J}_{M+1}^c \oplus R,\bullet}(0) = \boldsymbol{\mathcal{H}}_{\mathcal{J}_1^c,\bullet}(0) = \boldsymbol{0} \end{aligned}$$

where

$$\mathcal{J}_{M+1} \stackrel{\text{def}}{=} \mathcal{J}_1 \ominus R = (\mathcal{K}_1 \ominus R) \cup \mathcal{K}^M.$$

Therefore, using (28), we now have

$$\boldsymbol{\mathcal{H}}_{\mathcal{J}_{M+1},\bullet}\left(\frac{1}{T}\right) = \boldsymbol{\mathcal{G}}^\dagger_{\bullet,\mathcal{J}_{M+1}}\left(\frac{1}{T}\right), \ \boldsymbol{\mathcal{H}}_{\mathcal{J}_{M+1}^c,\bullet}\left(\frac{1}{T}\right) = \boldsymbol{0}. \tag{48}$$

To complete the proof, it suffices to construct a *continuous extension* $\boldsymbol{\mathcal{H}}(f)$ on $[0, 1/T]$ that satisfies (18), (47), and (48). With the intention of applying Lemma 1, define the following quantities:

$$\begin{aligned}\alpha &= \gamma_m\\ \boldsymbol{C}(f) &= \boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)\\ \boldsymbol{E}_\alpha &= \begin{pmatrix}\boldsymbol{\mathcal{H}}_{\mathcal{K}^m,\bullet}(\alpha)\\ \boldsymbol{\mathcal{H}}_{(\mathcal{J}_m\cup\mathcal{J}_{m+1})\setminus\mathcal{K}^m,\bullet}(\alpha)\end{pmatrix}\\ \beta &= \gamma_{m+1}\\ \boldsymbol{D}(f) &= \begin{pmatrix}\boldsymbol{I}_{|\mathcal{K}^m|}\\ \boldsymbol{0}\end{pmatrix}\\ \boldsymbol{E}_\beta &= \begin{pmatrix}\boldsymbol{\mathcal{H}}_{\mathcal{K}^m,\bullet}(\beta)\\ \boldsymbol{\mathcal{H}}_{(\mathcal{J}_m\cup\mathcal{J}_{m+1})\setminus\mathcal{K}^m,\bullet}(\beta)\end{pmatrix}.\end{aligned}$$

Observe that $\boldsymbol{C}(f) = \boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ has full column rank for $f \in [\gamma_m, \gamma_{m+1}]$. Moreover, using $\mathcal{K}^m \subseteq \mathcal{J}_m$ and $\mathcal{K}^m \subseteq \mathcal{J}_{m+1}$, it follows from (47) that

$$\begin{aligned}\boldsymbol{E}_\alpha\boldsymbol{C}(\alpha) &= \begin{pmatrix}\boldsymbol{I}_{|\mathcal{K}^m|}\\ \boldsymbol{0}\end{pmatrix} = \boldsymbol{D}(\alpha)\\ \boldsymbol{E}_\beta\boldsymbol{C}(\beta) &= \begin{pmatrix}\boldsymbol{I}_{|\mathcal{K}^m|}\\ \boldsymbol{0}\end{pmatrix} = \boldsymbol{D}(\beta).\end{aligned}$$

Thus, we have verified all the technical conditions required in Lemma 1, and we are guaranteed a continuous solution

$$\boldsymbol{E}(f) = \begin{pmatrix}\boldsymbol{\mathcal{H}}_{\mathcal{K}^m,\bullet}(f)\\ \boldsymbol{\mathcal{H}}_{(\mathcal{J}_m\cup\mathcal{J}_{m+1})\setminus\mathcal{K}^m,\bullet}(f)\end{pmatrix}$$

that meets the desired boundary conditions and satisfies

$$\begin{aligned}\boldsymbol{\mathcal{H}}_{\mathcal{K}^m,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f) &= \boldsymbol{I}_{|\mathcal{K}_f|} \text{ and}\\ \boldsymbol{\mathcal{H}}_{(\mathcal{J}_m\cup\mathcal{J}_{m+1})\setminus\mathcal{K}^m,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f) &= \boldsymbol{0}\end{aligned} \tag{49}$$

for $f \in [\gamma_m, \gamma_{m+1}]$. We also define

$$\boldsymbol{\mathcal{H}}_{(\mathcal{J}_m\cup\mathcal{J}_{m+1})^c,\bullet}(f) = \boldsymbol{0},\ f \in [\gamma_m, \gamma_{m+1}]. \tag{50}$$

Therefore, (49) and (50) provide us with a continuous extension for $\boldsymbol{\mathcal{H}}(f)$ on $[\gamma_m, \gamma_{m+1}]$ that satisfies (18) for each $m \in \mathcal{M}$ and, hence, for the entire interval [0, $1/T$]. Because $\boldsymbol{\mathcal{H}}(f)$ by construction also satisfies the boundary conditions (29), the continuity of $\boldsymbol{H}(f)$ follows by Proposition 2. ∎

Remark 1: A simple necessary condition for perfect reconstruction using continuous reconstruction filters is that $P \geq \max_m |\mathcal{J}_m|$.

Remark 2: Although the continuity of the entries of $\boldsymbol{G}(f)$ was essential in the above proof, it is not strictly necessary as it is possible to carefully construct examples where a continuous $\boldsymbol{H}(f)$ exists, even though $\boldsymbol{G}(f)$ may be discontinuous.

Example 6: The purpose of this example is to illustrate Theorem 2. Assume that $R = 2$ and $T = 4$ and that the input spectra have the same form as in Examples 2 and 4. Then, $\mathcal{K}_1 = \{0, 2, 6, 3\}$ and $\mathcal{K}_2 = \{0, 3\}$. In addition, the index sets defined in (41) are

$$\begin{aligned}\mathcal{J}_1 &= \mathcal{K}_1 \cup (\mathcal{K}_2 \oplus 2) = \{0, 2, 6, 3, 5\}\\ \mathcal{J}_2 &= \mathcal{K}_2 \cup \mathcal{K}_1 = \{0, 2, 3, 6\}.\end{aligned}$$

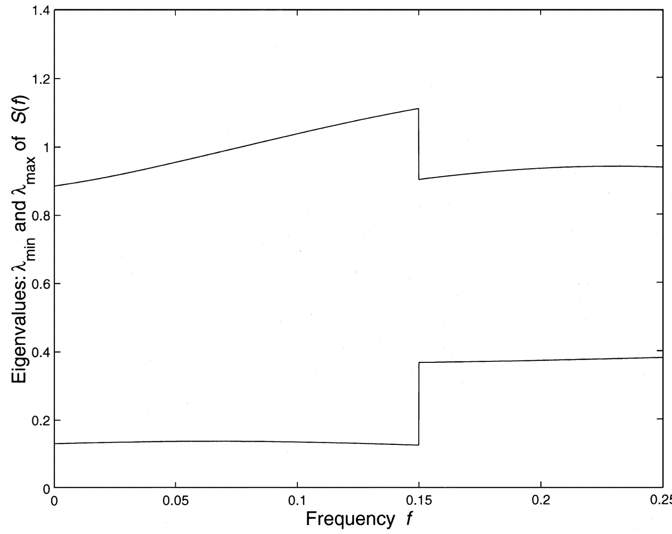

Fig. 4. Smallest and largest eigenvalues of $\boldsymbol{S}(f)$.

Hence, $P \geq \max_m |\mathcal{J}_m| = 5$ is necessary for the existence of a continuous $\boldsymbol{H}(f)$, and clearly, the transfer function matrix $\boldsymbol{G}(f)$ of Example 4 does not suffice. Therefore, let us append a new row beneath the last row of $\boldsymbol{G}(f)$, thereby making the MIMO channel a two-input five-output channel:

$$\boldsymbol{G}(f) = \begin{pmatrix}1 & 1\\ 1 & 1+e^{-j2\pi f}\\ e^{-j2\pi f} & 0.25+e^{-j4\pi f}\\ 1+0.5e^{-j2\pi f} & 1+e^{-j4\pi f}\\ 0.25+e^{-j4\pi f} & e^{-j2\pi f}\end{pmatrix}.$$

The rank condition in (39) holds because the matrix $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ of Example 4 has full column rank, and adding an extra row to $\boldsymbol{G}(f)$ (and hence to $\boldsymbol{\mathcal{G}}(f)$ as well) does not lower the column rank of $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$. Fig. 4 depicts the smallest and largest eigenvalues of the matrix

$$\boldsymbol{S}(f) = \boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)$$

as a function of frequency. Note that the discontinuities in these plots are expected because $\mathcal{K}_f(f)$ is piecewise constant with discontinuities at the cell boundaries, i.e., at $f = \gamma_1 = 0.15$ in this case. A numerical calculation yields the following frame bounds for the MIMO sampling scheme:

$$\begin{aligned}A &= \operatorname*{ess\,inf}_{f\in[0,1/T]} \lambda_{\min}(T\boldsymbol{S}(f)) = 0.1251\\ B &= \operatorname*{ess\,sup}_{f\in[0,1/T]} \lambda_{\max}(T\boldsymbol{S}(f)) = 1.1105.\end{aligned}$$

Hence, the condition number is $\sqrt{B/A} = 2.9790$. The other rank condition in (40), which needs to be verified at cell boundaries, also holds. Now, Theorem 2 guarantees the existence of a continuous filter matrix $\boldsymbol{H}(f)$ that achieves perfect reconstruction of the MIMO channel inputs.

The proof of Theorem 2 also provides, in principle, a method to construct a continuous reconstruction filter matrix $\boldsymbol{H}(f)$ when the conditions for its existence are satisfied. Specifically, fix $\boldsymbol{\mathcal{H}}(f)$ at the boundary points per (47) and (48), and then find a continuos solution $\boldsymbol{\mathcal{H}}(f)$ to the systems of linear equations (49) and (50) for $f \in [\gamma_m, \gamma_{m+1}], m \in \mathcal{M}$. The solution to these equations is, in general, nonunique, and a particular solution can be selected using additional criteria (for examples of such designs in the single channel case, see [33]). For example, the minimum norm solution will lead to minimum amplification of additive white noise on the sampled signals (due to e.g., quantization error). In any event, the final filter matrix $\boldsymbol{H}(f)$ is obtained from $\boldsymbol{\mathcal{H}}(f)$ via (14).

V. Conclusion

In this paper, we studied the uniform MIMO sampling problem. This scheme encompasses periodic nonuniform multicoset sampling, Papoulis' generalized sampling, and vector sampling schemes as a special cases. The MIMO problem is motivated by the problem of channel equalization from the sampled channel outputs. We presented necessary and sufficient conditions for perfect reconstruction of the signals or, equivalently, perfect inversion of the channel, when the input signals lie in the space of multiband signals with different band structures. We also presented the appropriate conditions for the existence of reconstruction filters with continuous frequency responses and specified them as solutions to a system of linear equations. The continuity property is important for the implementation of the reconstruction system because continuity allows arbitrarily close $\mathcal{H}^{\infty}$ approximation of the filter responses by sufficiently long FIR filters. We address the problem of reconstruction filter design using FIR filters in [34]. Finally, we demonstrated that in some cases, the sum sampling rate by multicoset sampling can equal the lower bound on the sampling density derived in [29]; however, the question of whether these lower bounds are generally achievable remains open.

Appendix
Proof of Lemma 1

Observe that $\boldsymbol{C}^H(f)\boldsymbol{C}(f)$ is nonsingular for all $f \in [\alpha, \beta]$ because rank $\boldsymbol{C}(f) = q$. In fact

$$\boldsymbol{C}^H(f)\boldsymbol{C}(f) > \epsilon^2 \boldsymbol{I}_q, \quad f \in [\alpha, \beta]$$

where ϵ is the minimum value of the smallest singular value of $\boldsymbol{C}(f)$ on $[\alpha, \beta]$:

$$\epsilon = \inf_{f \in [\alpha,\beta]} \sigma_{\min}(\boldsymbol{C}(f)) = \min_{f \in [\alpha,\beta]} \sigma_{\min}(\boldsymbol{C}(f)) > 0.$$

This is because $\boldsymbol{C}(f)$ is continuous, implying that $\sigma_{\min}(\boldsymbol{C}(f))$, which is also continuous on the compact set $[\alpha, \beta]$, attains its infimum. Therefore

$$\begin{aligned} \boldsymbol{C}^{\dagger}(f) &= \left(\boldsymbol{C}^H(f)\boldsymbol{C}(f)\right)^{-1}\boldsymbol{C}^H(f) \\ \boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))} &= \boldsymbol{C}(f)\left(\boldsymbol{C}^H(f)\boldsymbol{C}(f)\right)^{-1}\boldsymbol{C}^H(f) \end{aligned}$$

are also a continuous functions of f. Note that $\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))}$ is the orthogonal projection onto the range space of $\boldsymbol{C}(f)$. Define

$$\begin{aligned} \boldsymbol{E}_1 &:= \boldsymbol{E}_\alpha - \boldsymbol{D}(\alpha)\boldsymbol{C}^{\dagger}(\alpha) \\ \boldsymbol{E}_2 &:= \boldsymbol{E}_\beta - \boldsymbol{D}(\beta)\boldsymbol{C}^{\dagger}(\beta). \end{aligned}$$

It follows that $\boldsymbol{E}_1\boldsymbol{C}(\alpha) = \boldsymbol{E}_2\boldsymbol{C}(\beta) = \boldsymbol{0}$, implying that

$$\boldsymbol{E}_1\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\alpha))} = \boldsymbol{E}_2\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\beta))} = \boldsymbol{0}. \tag{A.1}$$

Now, take

$$\begin{aligned} \boldsymbol{E}(f) := {} & \boldsymbol{D}(f)\boldsymbol{C}^{\dagger}(f) \\ & + \left(\frac{(f-\alpha)}{(\beta-\alpha)}\boldsymbol{E}_2 + \frac{(\beta-f)}{(\beta-\alpha)}\boldsymbol{E}_1\right)\left(\boldsymbol{I}_q - \boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))}\right). \end{aligned}$$

This is a valid solution because it is continuous and meets the requirements

$$\begin{aligned} \boldsymbol{E}(f)\boldsymbol{C}(f) &= \boldsymbol{D}(f) + \left(\frac{(f-\alpha)}{(\beta-\alpha)}\boldsymbol{E}_2 + \frac{(\beta-f)}{(\beta-\alpha)}\boldsymbol{E}_1\right) \\ & \quad \times \left(\boldsymbol{C}(f) - \boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))}\boldsymbol{C}(f)\right) = \boldsymbol{D}(f) \\ \boldsymbol{E}(\alpha) &= \boldsymbol{D}(\alpha)\boldsymbol{C}^{\dagger}(\alpha) + \boldsymbol{E}_1 - \boldsymbol{E}_1\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\alpha))} = \boldsymbol{E}_\alpha \\ \boldsymbol{E}(\beta) &= \boldsymbol{D}(\beta)\boldsymbol{C}^{\dagger}(\beta) + \boldsymbol{E}_2 - \boldsymbol{E}_2\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\beta))} = \boldsymbol{E}_\beta. \end{aligned}$$

The last two equations follow from (A.1). ■

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Raman Venkataramani (S'97–M'01) received the B.Tech. degree from the Indian Institute of Technology, Madras, India, in 1995, the M.S. degree from Johns Hopkins University, Baltimore, MD, in 1997, and the Ph.D. degree from the University of Illinois at Urbana-Champaign in 2001, all in electrical engineering.

From November 2001 to December 2002, he was a member of technical staff at Bell Laboratories, Lucent Technologies, Murray Hill, NJ. From January 2003 to September 2003, he was a research associate at Harvard University, Cambridge, MA. He is now a post-doctoral researcher with Seagate Technology, Pittsburgh, PA. His interests include information theory, communications, statistical signal processing, signal sampling and interpolation, and inverse problems.

Yoram Bresler (F'99) received the B.Sc. (cum laude) and M.Sc. degrees from the Technion, Israel Institute of Technology, Haifa, Israel, in 1974 and 1981, respectively, and the Ph.D. degree from Stanford University, Stanford, CA, in 1986, all in electrical engineering.

From 1974 to 1979, he served as an electronics engineer in the Israeli Defense Force. From 1985 to 1987, he was a Research Associate with the Information Systems Laboratory, Stanford University, working on sensor array processing and medical imaging. In 1987, he joined the University of Illinois at Urbana-Champaign, where he is currently a Professor with the Department of Electrical and Computer Engineering and the Bioengineering Program and a Research Professor with the Coordinated Science Laboratory. From 1995 to 1996, he was a visiting Associate Professor at the Technion, Israel Institute of Technology, Haifa, Israel. His current research interests include multidimensional and statistical signal processing and their applications to inverse problems in imaging. He is on the editorial board of *Machine Vision and Applications*.

Dr. Bresler was an Associate Editor of the IEEE TRANSACTIONS ON IMAGE PROCESSING from 1992 to 1993 and a member of the IEEE Image and Multidimensional Signal Processing Technical Committee from 1994 to 1998. He is currently on the Awards Board of the IEEE Signal Processing Society (SPS). In 1988 and 1989, he received the Senior Paper Awards from the SPS, and a paper he coauthored with one of his students received the Young Author Award from the same society in 2002. He received a 1991 NSF Presidential Young Investigator Award, the Technion Fellowship in 1995, and the Xerox Senior Award for Faculty Research in 1998. He was named a University of Illinois Scholar in 1999 and appointed as an Associate at the Center for Advanced Study of the University for 2001 to 2002.