Filter Design for MIMO Sampling and Reconstruction

Raman Venkataramani, *Member, IEEE*, and Yoram Bresler, *Fellow, IEEE*

***Abstract*—We address the problem of finite impulse response (FIR) filter design for uniform multiple-input multiple-output (MIMO) sampling. This scheme encompasses Papoulis' generalized sampling and several nonuniform sampling schemes as special cases. The input signals are modeled as either continuous-time or discrete-time multiband input signals, with different band structures. We present conditions on the channel and the sampling rate that allow perfect inversion of the channel. Additionally, we provide a stronger set of conditions under which the reconstruction filters can be chosen to have frequency responses that are continuous. We also provide conditions for the existence of FIR perfect reconstruction filters, and when such do not exist, we address the optimal approximation of the ideal filters using FIR filters and a min-max l_2 end-to-end distortion criterion. The design problem is then reduced to a standard semi-infinite linear program. An example design of FIR reconstruction filters is given.**

***Index Terms*—Filter design, MIMO equalization, min-max criterion, multiband sampling, multichannel deconvolution, multiple source separation, multiple-input multiple-output (MIMO) channel, multirate signal processing, semi-infinite optimization, signal reconstruction.**

I. INTRODUCTION

MULTIPLE-INPUT multiple-output (MIMO) deconvolution or channel equalization involves the recovery of the inputs to a MIMO channel whose outputs can be observed and whose characteristics may either be known or unknown. The unknown inputs usually have overlapping spectra and, hence, share a common bandwidth. MIMO deconvolution is an important problem arising in numerous applications, including multisensor biomedical signals [5], [6], multitrack magnetic recording [7], multiple speaker (or other acoustic source) separation with microphone arrays [8], [9], geophysical data processing [10], and multichannel image restoration [11], [12]. MIMO deconvolution or equalization might also be used in communications applications such as multiuser or multiaccess wireless communications or telephone digital subscriber loops [1], [2], [4] when a simple linear preprocessor is desired, which does not use knowledge of the discrete nature of the digital communication signals (or only uses it to adapt the reconstruction filters).

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R. Venkataramani was with the Division of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138 USA. He is now with Seagate Technology, Pittsburgh, PA 15222 USA (e-mail: ramanv@ieee.org).

Y. Bresler is with the Coordinated Science Laboratory, Department of Electrical and Computer Engineering, University of Illinois, Urbana, IL 61801 USA (e-mail: ybresler@uiuc.edu).

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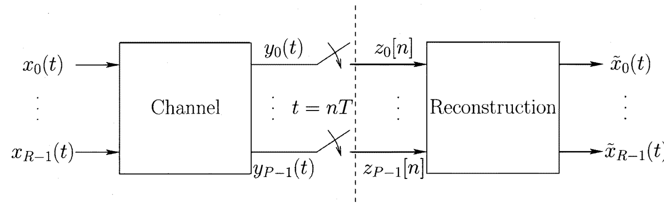

Fig. 1. Models for MIMO sampling and reconstruction. Only the sampled channel outputs $z_i[n]$ are observed, and the goal is to reconstruct the continuous-time channel inputs $x_i(t)$.

In practice, digital processing is used to perform the channel inversion. Consequently, the channel outputs need to be sampled prior to processing, and the objective is to reconstruct the channel inputs from the sampled output signals. The resulting channel inversion problem reduces to one in sampling theory that we call *MIMO sampling*. To focus on the sampling and reconstruction issues, we restrict our attention to the scenario of a linear time-invariant MIMO channel with known frequency response matrix. As appropriate in many applications, the input signals to the channel are assumed to be multiband signals, with possibly different band structures.

The continuous-time model for the MIMO channel and its reconstruction [13] is illustrated in Fig. 1. The linear time-invariant channel has R multiband inputs $x_r(t), r = 0, \ldots, R-1$, and P outputs $y_p(t)$, $p = 0, \ldots, P-1$. Only the sampled channel outputs $z_p[n] = y_p(nT)$ are observed, and the goal is to reconstruct the continuous-time channel inputs $x_r(t)$.

Because the signal processing is usually done digitally, it is convenient to consider an equivalent discrete-time model for the system. Since the channel inputs have bounded spectral supports, we can convert the continuous-time channel model into the linear time-invariant discrete-time model in Fig. 2, where the sequences $x_r[k] = x_r(kT_s)$ represent samples of the underlying continuous-time signals taken at a sufficiently high rate. For example, $1/T_s$ can be chosen equal to or larger than the highest of the Nyquist rates of any of the individual continuous-time inputs $x_r(t), r = 0, \ldots, R-1$. The discrete-time channel is represented by its frequency response matrix $\boldsymbol{G}[\nu]$ relating the inputs to outputs. Downsampling the outputs in the discrete-time model by an integer factor $L > 0$ then produces the observed outputs $z_p[n] = y_p[Ln] = y_p(nLT_s) = y_p(nT)$, which coincide with the sampled outputs of the continuous-time channel,[1] thus completing the equivalence between the channel models in Figs. 1 and 2. The role of downsampling can also be understood in a purely discrete-time context; in general, the spectral band structures may allow us to reconstruct the inputs using only a subset of the output samples. The discrete-time reconstruction block, which is depicted in Fig. 3, produces estimates $\tilde{x}_r[k]$ of

[1] L can be taken as an integer without loss of generality for any given T in Fig. 1 by choosing $T_s = T/L$.

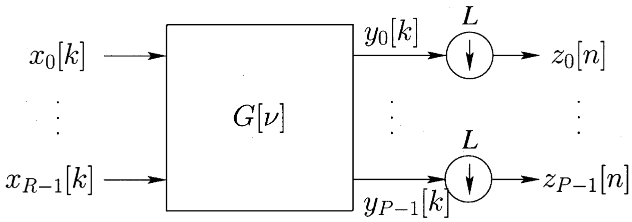

Fig. 2. Discrete-time model for the MIMO channel.

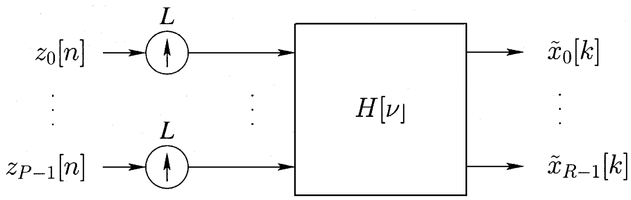

Fig. 3. Discrete-time model for MIMO reconstruction.

the input signals from the observed signals $z_p[n]$. The continuous-time inputs can finally be recovered from the discrete-time sequences $\tilde{x}_r[k]$ using a bank of conventional D/A converters.

In this paper, we present sufficient conditions for perfect reconstruction in the discrete-time model with uniform subsampling (see [15] and [16] for necessary density conditions on arbitrary nonuniform sampling) and conditions and a solution to the related filter design problem. We will consider only uniform subsampling of the channel outputs. This sampling scheme is fairly general and subsumes periodic nonuniform subsampling of the MIMO outputs as a special case of uniform subsampling applied to a hypothetical channel with more outputs [13]. Furthermore, several familiar sampling schemes can be viewed as special cases of MIMO sampling. For example, in Papoulis' generalized sampling [17], a single lowpass input signal is passed through a bank of M filters, and the outputs are sampled at $1/M$th the Nyquist rate of the signal. This fits in our framework as a single-input multiple-output sampling problem, i.e., $R = 1$. Additionally, if the channel filters are pure delays, we obtain multicoset or periodic nonuniform sampling of the input signal, which has been widely studied [18]–[30], as it allows the approach of the Landau minimum sampling rate for multiband signals [31]. Seidner and Feder [32] provide a natural generalization of Papoulis' sampling expansions for a vector input with its components bandlimited to $[-B, B]$. Clearly, their sampling scheme is also a special case of MIMO sampling.

We studied the continuous-time MIMO sampling problem and presented necessary and sufficient conditions for perfect stable reconstruction of the channel inputs from uniform sampling of the outputs in [13]. Importantly, we demonstrated how to achieve stable sampling and reconstruction at rates lower than the Nyquist rate of each of the individual inputs and, in some cases, even at a combined average rate lower than the Landau rate of each of the individual inputs. This provides motivation for using the MIMO sampling theory to design and implement MIMO deconvolution and source separation systems.

In this paper, we examine the related problem of finite impulse response (FIR) filter design for MIMO reconstruction filters. Whereas [13] only demonstrates the existence of ideal (i.e., unrealizable) filters for stable perfect reconstruction subject to appropriate conditions on the channel and sampling rates, in this paper, we address the practical problem of implementing the reconstruction system using FIR filters. We provide conditions for the existence of FIR perfect reconstruction filters, and when such do not exist, we address the optimal approximation of the ideal filters using FIR filters and a min-max l_2 reconstruction error criterion. We formulate the design problem as a semi-infinite linear program. Semi-infinite formulations have been successfully applied to other multirate filter design problems [33], [34] and solved using standard techniques [35]. Our FIR filter design formulation is fairly general and can be used to design the interpolation filters for those generalized sampling schemes discussed above.

The paper is organized as follows. Section II contains some basic notation and definitions. In Section III, we present discrete-time models for the channel and reconstruction block. The channel inputs are modeled as multiband signals. In Section IV-A, we present discrete-time versions of the results derived in [13]. In particular, we specify necessary and sufficient conditions for the existence of reconstruction filters that are continuous in the frequency domain. This property is important in the context of FIR filter design, as we elaborate upon later. Finally, in Section V, we discuss the problem of FIR reconstruction filter design for the MIMO sampling problem. We formulate a cost function in terms of the filter coefficients. Minimizing the cost produces the optimal filter coefficients. The problem may be recast as a semi-infinite linear program. We present two design examples: one for multicoset sampling and another for MIMO sampling with two inputs.

II. Preliminaries

We begin with some basic definitions and notation. Denote the discrete-time Fourier transform of a $x[n] \in l^2$ by the periodic function

$$X[\nu] = \sum_{k \in \mathbb{Z}} x[k] e^{-j2\pi\nu k}.$$

In general, we denote time signals (either scalar-valued or vector-valued) using lower-case letters and their Fourier transforms by the corresponding upper-case letters. Denote the class of complex-valued, finite-energy discrete-time signals bandlimited to the set of frequencies $\mathcal{F} \subseteq [0, 1)$ by

$$\mathcal{B}(\mathcal{F}) = \{x[k] \in l^2 : X[\nu] = 0, \ \forall \nu \in [0, 1) \cap \mathcal{F}^c\}. \quad (1)$$

We denote the class of complex-valued matrices of size $M \times N$ by $\mathbb{C}^{M \times N}$, the conjugate-transpose of $\boldsymbol{A}$ by $\boldsymbol{A}^H$, its pseudo inverse by $\boldsymbol{A}^\dagger$, and its range space by $\mathcal{R}(\boldsymbol{A})$. For a given matrix $\boldsymbol{A}$, let $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$ denote the submatrix of $\boldsymbol{A}$ corresponding to rows indexed by the set $\mathcal{R}$ and columns by the set $\mathcal{C}$. The quantity $\boldsymbol{A}_{\bullet,\mathcal{C}}$ denotes a submatrix formed by keeping all rows of $\boldsymbol{A}$ but only columns indexed by $\mathcal{C}$, whereas $\boldsymbol{A}_{\mathcal{R},\bullet}$ denotes the submatrix formed by retaining rows indexed by $\mathcal{R}$ and all columns. We use a similar notation for vectors. Hence, $\boldsymbol{X}_{\mathcal{R}}$ is the subvector of $\boldsymbol{X}$ corresponding to rows indexed by $\mathcal{R}$. We always apply the subscripts of a matrix before the superscript. Therefore, $\boldsymbol{A}^H_{\mathcal{R},\mathcal{C}}$ is the conjugate-transpose of $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$. When dealing with singleton index sets $\mathcal{R} = \{r\}$ or $\mathcal{C} = \{c\}$, we omit the curly braces for readability. Therefore, $\boldsymbol{A}_{r,\bullet}$ and $\boldsymbol{A}_{\bullet,c}$ are the rth row and the cth column of $\boldsymbol{A}$, respectively. For convenience, we always number the rows and columns of a finite-size matrix starting from 0. For infinite-size matrices, the row and column indices range over $\mathbb{Z}$. As a result of the above notation, we have the following straightforward proposition that is used later.

Proposition 1: Suppose that $\boldsymbol{X} \in \mathbb{C}^L$ and that $\boldsymbol{I}$ is the $L \times L$ identity matrix. Then, $\boldsymbol{X}_{\mathcal{K}} = \boldsymbol{I}_{\mathcal{K},\bullet}\boldsymbol{X}$ for all $\mathcal{K} \subseteq \{0,\ldots,L-1\}$. Additionally, if $\boldsymbol{X}_{\mathcal{K}^c} = \mathbf{0}$, where $\mathcal{K}^c$ is the complement of $\mathcal{K}$, then $\boldsymbol{X} = \boldsymbol{I}_{\bullet,\mathcal{K}}\boldsymbol{X}_{\mathcal{K}}$.

The identity matrix of size $N \times N$ is denoted by $\boldsymbol{I}_N$ and the zero matrix by $\mathbf{0}$. Finally, suppose that $\mathcal{S}$ is a subset of $\mathbb{R}$ or $\mathbb{Z}$, and a is an element of $\mathbb{R}$ or $\mathbb{Z}$; then

$$\mathcal{S} \oplus a = \{s + a : s \in \mathcal{S}\}, \quad \mathcal{S} \ominus a = \{s - a : s \in \mathcal{S}\}$$
$$a\mathcal{S} = \{as : s \in \mathcal{S}\}, \quad \mathcal{S} \bmod a = \{s \bmod a : s \in \mathcal{S}\}$$

denote the positive and negative translations, scaling, and the modulus of $\mathcal{S}$ by a, respectively.

III. Sampling and Reconstruction Models

Fig. 2 depicts a discrete-time MIMO channel with inputs $\{x_0[k],\ldots,x_{R-1}[k]\}$ and outputs $\{y_0[k],\ldots,y_{P-1}[k]\}$. Let $\mathcal{R} = \{0,1,\ldots,R-1\}$ and $\mathcal{P} = \{0,1,\ldots,P-1\}$ denote index sets for the channel inputs and outputs. We model $x_r[k]$, $r \in \mathcal{R}$ as multiband signals $x_r[k] \in \mathcal{B}(\mathcal{F}_r)$, where the *spectral support* $\mathcal{F}_r \subseteq [0,1)$ is a finite union of disjoint intervals:

$$\mathcal{F}_r = \bigcup_{n=1}^{N_r} [a_{rn}, b_{rn}), \quad a_{r1} < b_{r1} < a_{r2} < \ldots < a_{rN_r} < b_{rN_r}. \tag{2}$$

Let the channel inputs and outputs be expressed in vector form as

$$\boldsymbol{x}[k] = \left(x_0[k] \;\; x_1[k] \;\; \cdots \;\; x_{R-1}[k]\right)^T$$
$$\boldsymbol{y}[k] = \left(y_0[k] \;\; y_1[k] \;\; \cdots \;\; y_{P-1}[k]\right)^T.$$

The MIMO channel is modeled as a linear shift-invariant system with channel impulse response matrix $\boldsymbol{g}[k] \in \mathbb{C}^{P\times R}$. The entry $g_{rp}[k]$ corresponds to the filter that sends the pth input to the rth output. The input-output relations in the time and frequency domain are thus

$$\boldsymbol{y}[k] = \boldsymbol{g} * \boldsymbol{x}[k] = \sum_{k' \in \mathbb{Z}} \boldsymbol{g}[k-k']\boldsymbol{x}[k']$$
$$\boldsymbol{Y}[\nu] = \boldsymbol{G}[\nu]\boldsymbol{X}[\nu] \tag{3}$$

where $*$ denotes convolution, and $\boldsymbol{X}[\nu]$, $\boldsymbol{Y}[\nu]$, and $\boldsymbol{G}[\nu]$ are the discrete-time Fourier transforms of $\boldsymbol{x}[k]$, $\boldsymbol{y}[k]$, and $\boldsymbol{g}[k]$, respectively. We call $\boldsymbol{G}[\nu]$ the *channel frequency response matrix*. The channel outputs are uniformly subsampled by a factor of L, and the resulting vector sequence is denoted by $\boldsymbol{z}[n] = \boldsymbol{y}[nL]$, $n \in \mathbb{Z}$. Using (3), we now have

$$\boldsymbol{Z}[\nu] = \frac{1}{L}\sum_{l=0}^{L-1} \boldsymbol{Y}\left[\frac{\nu+l}{L}\right] = \frac{1}{L}\sum_{l=0}^{L-1} \boldsymbol{G}\left[\frac{\nu+l}{L}\right]\boldsymbol{X}\left[\frac{\nu+l}{L}\right] \tag{4}$$

for $\nu \in [0,1)$, which essentialy describes the input–output relation of a filterbank with vector-valued inputs.

We model the reconstruction block as follows:

$$\tilde{\boldsymbol{x}}[k] = \sum_{n\in\mathbb{Z}} \boldsymbol{h}[k-nL]\boldsymbol{z}[n] \tag{5}$$

where $\boldsymbol{h}[k] \in \mathbb{C}^{R\times P}$ is the impulse response matrix of the reconstruction filter.

From (5), it is obvious that the entire system consisting of the channel, subsampling, and reconstruction is invariant to time-shifts by any multiple of L, i.e.,

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} \Longrightarrow \boldsymbol{x}[\cdot - nL] \to \tilde{\boldsymbol{x}}[\cdot - nL], \quad \forall n \in \mathbb{Z}.$$

Conversely, (5) describes the most general linear transformation that allows this invariance. The discrete-time Fourier transform of (5) is $\tilde{\boldsymbol{X}}[\nu] = \boldsymbol{H}[\nu]\boldsymbol{Z}[L\nu]$ for $\nu \in [0,1)$, where $\boldsymbol{H}[\nu]$, which is the discrete-time Fourier transform of $\boldsymbol{h}[n]$, is called the *reconstruction filter frequency response matrix*. Since $\boldsymbol{Z}[\nu]$ is a periodic function, we have

$$\tilde{\boldsymbol{X}}\left[\nu + \frac{l'}{L}\right] = \boldsymbol{H}\left[\nu + \frac{l'}{L}\right]\boldsymbol{Z}[L\nu], \quad l' \in \mathbb{Z},\ \nu \in \left[0, \frac{1}{L}\right). \tag{6}$$

Let $\mathcal{L} = \{0,1,\ldots,L-1\}$. We can now write (4) and (6) compactly as

$$Z[L\nu] = \boldsymbol{\mathcal{G}}[\nu]\boldsymbol{\mathcal{X}}[\nu] \tag{7}$$
$$\tilde{\boldsymbol{\mathcal{X}}}[\nu] = \boldsymbol{\mathcal{H}}[\nu]\boldsymbol{Z}[L\nu] \tag{8}$$

for $\nu \in [0, 1/L)$, where $\boldsymbol{\mathcal{X}}[\nu] \in \mathbb{C}^{LR}$ is defined as

$$\mathcal{X}_{Rl+r}[\nu] = X_r\left[\nu + \frac{l}{L}\right], \quad (r,l) \in \mathcal{R}\times\mathcal{L} \tag{9}$$

and $\tilde{\boldsymbol{\mathcal{X}}}[\nu] \in \mathbb{C}^{LR}$ is defined analogously, whereas $\boldsymbol{\mathcal{G}}[\nu] \in \mathbb{C}^{P\times RL}$ and $\boldsymbol{\mathcal{H}}[\nu] \in \mathbb{C}^{RL\times P}$ are the *modulated channel and reconstruction frequency response matrices*, which are defined as

$$\mathcal{G}_{p,Rl+r}[\nu] = \frac{1}{L} G_{pr}\left[\nu + \frac{l}{L}\right], \quad (p,r,l) \in \mathcal{P}\times\mathcal{R}\times\mathcal{L} \tag{10}$$
$$\mathcal{H}_{Rl+r,p}[\nu] = H_{rp}\left[\nu + \frac{l}{L}\right], \quad (p,r,l) \in \mathcal{P}\times\mathcal{R}\times\mathcal{L}. \tag{11}$$

In the next section, we provide precise conditions for stable reconstruction of the channel inputs from the subsampled output sequences. In particular, for FIR implementation reasons, we are interested in a reconstruction filter matrix $\boldsymbol{H}[\nu]$ whose entries are continuous in ν. Specifically, the continuity guarantees that the approximation error resulting from the FIR implementation can be made arbitrarily small by choosing sufficiently long FIR filters. This point will be elaborated upon later.

IV. Conditions for Perfect Reconstruction

A. General Case

In this section, we present the condition for perfect reconstruction from the MIMO channel outputs in the discrete-time setting. More specifically, we provide conditions on the channel frequency response matrix $\boldsymbol{G}[\nu]$ that guarantee stable reconstruction of the inputs *with*/*without* the continuity requirement on the reconstruction filter matrix $\boldsymbol{H}[\nu]$. These results are discrete-time versions of their continuous-time counterparts in [13].

Let $\mathbb{H} = \mathcal{B}(\mathcal{F}_1) \times \cdots \times \mathcal{B}(\mathcal{F}_R)$ denote the class of inputs to the MIMO channel. Then, $\mathbb{H}$ is a Hilbert space equipped with the inner product $\langle \boldsymbol{x}, \boldsymbol{w} \rangle = \sum_{n\in\mathbb{Z}} \boldsymbol{w}^H[n]\boldsymbol{x}[n]$ for all $\boldsymbol{x}, \boldsymbol{w} \in \mathbb{H}$. Naturally, $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$ is the the norm of $\boldsymbol{x} \in \mathbb{H}$. We first review an important notion called *stability* of MIMO sampling [13], [15].

Definition 1: The MIMO sampling scheme is called stable if there exist constants $A, B > 0$ such that

$$A\|\boldsymbol{x}\|^2 \le \sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2 \le B\|\boldsymbol{x}\|^2, \quad \forall \boldsymbol{x} \in \mathbb{H}. \tag{12}$$

The implication of (12) is that we can reconstruct $\boldsymbol{x}$ from $\boldsymbol{z}[n]$ stably in the sense that small perturbations in the inputs or the channel output samples cannot cause large errors in the reconstructed outputs. Note that in the absence of the MIMO channel, i.e., $\boldsymbol{y}[k] = \boldsymbol{x}[k]$, the condition (12) reduces to the standard definition of stable sampling because $z[n] = \boldsymbol{y}[nL] = \boldsymbol{x}[nL]$ is a subsequence of $\boldsymbol{x}[k]$.

The *condition number* of the sampling scheme is $K = \sqrt{B/A} \ge 1$, and it bounds the amplification of the normalized 2-norm of the error due to the reconstruction filters [13]. In particular, the notion of stable sampling may be expressed as a frame-theoretic condition. See [36] for more about frames and [37] and [38] for their application in sampling theory.

Now, define the following index sets

$$\begin{aligned}\mathcal{K}_\nu &= \left\{Rl + r : (r,l) \in \mathcal{R} \times \mathcal{L} \text{ and } \nu + \frac{l}{L} \in \mathcal{F}_r\right\}\\ \mathcal{K}_\nu^c &= \{0,\ldots,RL-1\}\backslash\mathcal{K}_\nu\end{aligned} \tag{13}$$

for $\nu \in [0, 1/L]$. Just as in the continuous-time case in [13], we can decompose the interval $[0, 1/L]$ into a union of intervals where $\mathcal{K}_\nu$ is piecewise constant.

Proposition 2: Suppose that sets $\mathcal{F}_r$, $r \in \mathcal{R}$ have multiband structure, as defined in (2). Then, there exists a collection of disjoint intervals $\mathcal{I}_m$ and sets $\mathcal{K}^m$, $m = 1,\ldots,M$ such that

$$\bigcup_{m=1}^{M} \mathcal{I}_m = \left[0, \frac{1}{L}\right) \quad \text{and} \quad \mathcal{K}_\nu = \mathcal{K}^m, \quad \forall \nu \in \mathcal{I}_m.$$

This result is easily demonstrated by using an argument very similar to the one in [29].

We write $\mathcal{I}_m = [\gamma_m, \gamma_{m+1})$ for $m \in \mathcal{M} = \{1,\ldots M\}$, with $\gamma_1 < \gamma_2 < \ldots < \gamma_{M+1}$, such that $\gamma_1 = 0$ and $\gamma_{M+1} = 1/L$. For convenience, we also define

$$q_m \stackrel{\text{def}}{=} |\mathcal{K}^m|.$$

Equation (13) implies that all nonzero entries of $\boldsymbol{\mathcal{X}}[\nu]$ are captured in $\boldsymbol{\mathcal{X}}_{\mathcal{K}_\nu}[\nu]$. Hence, from (7) and (8), we conclude that for perfect reconstruction, $\boldsymbol{\mathcal{H}}_{\mathcal{K}_\nu,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_\nu}[\nu] = \boldsymbol{I}_{|\mathcal{K}_\nu|}$ and $\boldsymbol{\mathcal{H}}_{\mathcal{K}_\nu^c,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_\nu}[\nu] = \boldsymbol{0}$ must hold almost everywhere (a.e.). This characterization of $\boldsymbol{\mathcal{H}}$ that provides perfect reconstruction can be written compactly as

$$\boldsymbol{\mathcal{H}}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_\nu}[\nu] = \boldsymbol{I}_{\bullet,\mathcal{K}_\nu}, \quad \text{a.e.} \tag{14}$$

where $\boldsymbol{I}$ is the $RL \times RL$ identity matrix. Since $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_\nu}[\nu] \in \mathbb{C}^{P\times|\mathcal{K}_\nu|}$, (14) requires that $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_\nu}(\nu)$ have full column rank a.e.

As in the continuous-time case [13], it can be easily verified that $\boldsymbol{H}[\nu]$ is continuous if and only if $\boldsymbol{\mathcal{H}}[\nu]$ is continuous on $[0, 1/L]$, and the following periodicity conditions hold:

$$\boldsymbol{\mathcal{H}}_{\mathcal{K},\bullet}\left[\frac{1}{L} + \nu\right] = \boldsymbol{\mathcal{H}}_{\mathcal{K}',\bullet}[\nu] \tag{15}$$

for all $\mathcal{K} \subseteq \{0,\ldots,RL-1\}$, where $\mathcal{K}' = (\mathcal{K} \oplus R)\text{mod}RL$. As we will see later, in order to achieve continuity of $\boldsymbol{H}[\nu]$, it is convenient to impose continuity on $\boldsymbol{G}[\nu]$, and this produces a similar condition on $\boldsymbol{\mathcal{G}}[\nu]$. Specifically, if $G_{pr}[\nu]$ is continuous on $\overline{\mathcal{F}}_r$ (the closure of $\mathcal{F}_r$), then $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu]$ is continuous on $\overline{\mathcal{I}}_m$, and

$$\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}\left[\frac{1}{L} + \nu\right] = \boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}'}[\nu] \tag{16}$$

for all $\mathcal{K} \subseteq \{0,\ldots,RL-1\}$.

The following theorem, which is the discrete-time version of similar results presented in [13], provides precise conditions for stable and perfect reconstruction of the channel inputs. We do not prove these conditions, as they can be deduced in a manner very similar to their continuous-time counterparts in [13]. Before stating the results, we point out that ess inf and ess sup denote the essential infimum and supremum respectively, i.e.,

$$\begin{aligned}\text{ess inf} g(t) &= \sup\{\gamma : g(t) \ge \gamma \text{ a.e.}\}\\ \text{ess sup} g(t) &= \inf\{\gamma : g(t) \le \gamma \text{ a.e.}\}\end{aligned}$$

for any real function g.

Theorem 1: Suppose that $\boldsymbol{G}[\nu]$ is such that $G_{pr}[\nu]$ is continuous for $\nu \in \overline{\mathcal{F}}_r$, and $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu]$ has full column rank for all $m \in \mathcal{M}$, $\nu \in \overline{\mathcal{I}}_m = [\gamma_m, \gamma_{m+1}]$; then, the MIMO sampling scheme is stable, and the stability bounds are given by

$$A = L \inf_{\nu\in[0,1/L]} \lambda_{\min}\left(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_\nu}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_\nu}[\nu]\right) \tag{17}$$

$$B = L \sup_{\nu\in[0,1/L]} \lambda_{\max}\left(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_\nu}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_\nu}[\nu]\right). \tag{18}$$

Furthermore, the existence of a continuous reconstruction filter matrix $\boldsymbol{H}[\nu]$ is guaranteed if and only if

$$\begin{aligned}\text{rank}\left(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu]\right) &= |\mathcal{K}^m|\\ \forall \nu \in \text{int}\,\mathcal{I}_m &= (\gamma_m, \gamma_{m+1})\end{aligned} \tag{19}$$

$$\text{rank}\left(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{J}_m}([\gamma_m])\right) = |\mathcal{J}_m|, \quad m \in \mathcal{M} \tag{20}$$

where

$$\begin{aligned}\mathcal{J}_m &= \mathcal{K}^m \cup \mathcal{K}^{m-1}, \quad m = 2,\ldots,M\\ \mathcal{J}_1 &= \mathcal{K}_1 \cup ((\mathcal{K}^M \oplus R) \text{ mod } RL).\end{aligned} \tag{21}$$

Theorem 1 provides conditions for stable reconstruction and continuity of at least one solution $\boldsymbol{H}[\nu]$ whose corresponding modulated frequency response matrix satisfies (14). [Note that, in general, (14) does not have a unique solution because $\mathcal{K}_\nu \le P$.] The continuity requirement is desirable from the viewpoint of implementation, as we see in Section V. A simple necessary condition for perfect reconstruction using continuous reconstruction filters is provided by the following.

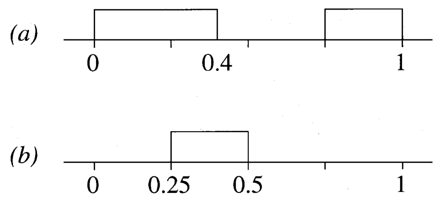

Fig. 4. (a) Spectral support of (a) $X_0[\nu]$ and (b) $X_1[\nu]$ for Example 1.

Corollary 1: Perfect reconstruction using continuous reconstruction filters is only possible if $P \geq \max_m |\mathcal{J}_m|$.

Note that when all inputs have identical band structure $\mathcal{F}_r = \mathcal{F},\ \forall r \in \mathcal{R}$, the necessary condition of Corollary 1 reduces to the familiar condition $P \geq RL$. However, in general, the spectral band structure of the inputs can be such that a smaller P (even the bound in the corollary) suffices. This is illustrated in the examples.

In Theorem 1, the assumption that $\boldsymbol{G}[\nu]$ is continuous in ν is made for convenience; it is possible for continuous perfect reconstruction filter matrix $\boldsymbol{H}[\nu]$ to exist, despite the lack of continuity of $\boldsymbol{G}[\nu]$. However, this is rare, and the conditions in the general case are cumbersome. On the other hand, perfect reconstruction is not possible when $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu]$ is rank deficient. In this case, one could compute the solution that minimizes the least squares approximation error. We do not consider this problem here; however, see [39] for related work.

Example 1: Consider a MIMO channel with $R = 2$ inputs and $P = 5$ outputs. Suppose that the inputs $x_0[k]$ and $x_1[k]$ have spectral supports illustrated in Fig. 4, namely

$$\mathcal{F}_0 = [0, 0.4) \cup [0.75, 1.0) \text{ and } \mathcal{F}_1 = [0.25, 0.5)$$

and $\boldsymbol{G}[\nu]$ denotes the following continuous channel transfer function matrix with $P = 5$ outputs:

$$\boldsymbol{G}[\nu] = \begin{pmatrix} 1 & 1 \\ 1 & 1 + z^{-1} \\ z^{-1} & 0.25 + z^{-2} \\ 1 + 0.5z^{-1} & 1 + z^{-2} \\ 0.25 + z^{-2} & z^{-1} \end{pmatrix}, \text{ where } z = e^{j2\pi\nu}. \tag{22}$$

Let $L = 4$ be the subsampling factor. For this choice, we have $M = 2$, $\mathcal{I}_1 = [0, 0.15)$, and $\mathcal{I}_2 = [0.15, 0.25)$. Using (13) and (21), it is easy to check that

$$\begin{aligned} \mathcal{K}_1 &= \{0, 2, 3, 6\} \\ \mathcal{K}_2 &= \{0, 3, 6\} \\ \mathcal{J}_1 &= \mathcal{K}_1 \cup ((\mathcal{K}_2 \oplus 2) \bmod 8) = \{0, 2, 3, 5, 6\} \\ \mathcal{J}_2 &= \mathcal{K}_2 \cup \mathcal{K}_1 = \{0, 2, 3, 6\}. \end{aligned}$$

Hence, by Theorem 1, $P \geq \max_m |\mathcal{K}^m| = 4$ is required for the existence of a reconstruction filter matrix $\boldsymbol{H}[\nu]$ that achieves perfect reconstruction. If we also require that the filters be continuous, Corollary 1 states that $P \geq \max_n |\mathcal{J}_n| = 5$ is necessary. It can be verified numerically that (19) and (20) in Theorem 1 are satisfied for the channel (22). Hence, we are guaranteed the existence of a perfect reconstruction filter matrix $\boldsymbol{H}[\nu]$ that is continuous in ν. This example, with $P = 5$ and $RL = 8$, also illustrates that owing to the band structure of the inputs, perfect reconstruction is possible even with $P < RL$.

B. FIR Reconstruction

Under certain conditions, perfect reconstruction is realizable using FIR filters. We say that a filter is FIR if its impulse response has a finite number of nonzero terms. An FIR filter need not be causal but can be made causal by adding a finite delay. The obvious advantage of having FIR reconstruction filter is simplicity of implementation.

For convenience, we consider only FIR channels because they can be parametrized using a finite number of parameters. We will see in Theorem 2 that the FIR assumption on the channel does not guarantee perfect reconstruction using FIR filters unless the channel satisfies additional conditions. (More generally, a channel with a rational transfer function matrix can also be described using a finite number of parameters, but we do not consider this case here.) Let

$$\boldsymbol{G}^\star[z] = \sum_{k \in \mathbb{Z}} \boldsymbol{g}[k] z^{-k}$$

denote the Z-transform of $\boldsymbol{g}[k]$. We use the superscript "$\star$" here to distinguish between the Z-transform and the discrete-time Fourier transform $\boldsymbol{G}[\nu]$ (transfer function and frequency response matrices, respectively). Then, clearly

$$\boldsymbol{G}[\nu] = \boldsymbol{G}^\star[\exp(j2\pi\nu)], \quad \nu \in \mathbb{R}.$$

Let $\boldsymbol{H}^\star[z]$ denote the Z-transform of $\boldsymbol{h}[k]$. Finally, let $\boldsymbol{\mathcal{G}}^\star[z]$ and $\boldsymbol{\mathcal{H}}^\star[z]$ be the Z-domain analogs of the modulated frequency response matrices $\boldsymbol{\mathcal{G}}[\nu]$ and $\boldsymbol{\mathcal{H}}[\nu]$. From (10) and (11), we see that $\mathcal{G}^\star_{p,Rl+r}[z] = \frac{1}{L} G^\star_{pr}[ze^{j2\pi l/L}]$ and $\mathcal{H}^\star_{Rl+r,p}[z] = H^\star_{rp}[ze^{j2\pi l/L}]$ for $(p, r, l) \in \mathcal{P} \times \mathcal{R} \times \mathcal{L}$. Accordingly, we call $\boldsymbol{\mathcal{G}}^\star[z]$ and $\boldsymbol{\mathcal{H}}^\star[z]$ the modulated channel and reconstruction transfer function matrices.

Theorem 2: Suppose that the channel impulse response $\boldsymbol{g}[k]$ is FIR, and let

$$\mathcal{K} = \bigcup_{l=0}^{L-1} \bigcup_{m=1}^{M} [(\mathcal{K}^m \oplus Rl) \bmod RL] \tag{23}$$

and $Q = |\mathcal{K}|$. Then, perfect reconstruction using an FIR reconstruction filter matrix $\boldsymbol{H}^\star[z]$ is possible if and only if $P \geq Q$, *and the* $Q \times Q$ minors of $\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z]$ have no zero common to all except $z = 0$ or $z = \infty$.

Proof: The coprimeness condition on the minors guarantees the existence of an FIR $\boldsymbol{\mathcal{H}}^\star_{\mathcal{K},\bullet}[z]$ such that $\boldsymbol{\mathcal{H}}^\star_{\mathcal{K},\bullet}[z]\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z] = \boldsymbol{I}$ and the periodicity conditions (15) and (16) hold. This is a standard result that can be proved using Bezout's identity [40]–[42]. Furthermore, letting $\boldsymbol{\mathcal{H}}^\star_{\mathcal{K}^c,\bullet}[z] = \boldsymbol{0}$, we also obviously have $\boldsymbol{\mathcal{H}}^\star_{\mathcal{K}^c,\bullet}[z]\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z] = \boldsymbol{0}$. Combining the last two results, we obtain $\boldsymbol{\mathcal{H}}^\star[z]\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z] = \boldsymbol{I}_{\bullet,\mathcal{K}}$. Since $\mathcal{K}^m \subseteq \mathcal{K}$, it follows that

$$\boldsymbol{\mathcal{H}}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu] = \boldsymbol{I}_{\bullet,\mathcal{K}^m}, \quad \forall \nu \in \mathcal{I}_m \tag{24}$$

which is essentially equivalent to (14). Thus, we have found an FIR realization of perfect reconstruction filters.

Conversely, suppose that $\boldsymbol{h}[k]$ is an FIR filter matrix achieving prefect reconstruction. Then, $\boldsymbol{\mathcal{G}}[\nu]$ and $\boldsymbol{\mathcal{H}}[\nu]$ are analytic if ν is viewed as a complex variable since both $\boldsymbol{g}[k]$ and $\boldsymbol{h}[k]$ are finite sequences. Because, by (24), the analytic function $\boldsymbol{\mathcal{H}}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu] - \boldsymbol{I}_{\bullet,\mathcal{K}^m}$ vanishes on an interval, it follows that it must vanish everywhere, and

$$\boldsymbol{\mathcal{H}}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu] = \boldsymbol{I}_{\bullet,\mathcal{K}^m} \tag{25}$$

holds for all $\nu \in \mathbb{R}$ and $m \in \mathcal{M}$, rather than just $\nu \in \mathcal{I}_m$. For fixed $m \in \mathcal{M}$ and $l \in \mathcal{L}$, define $(\mathcal{K}^{m\prime} = \mathcal{K}_m \oplus Rl)\text{mod}RL$ and $\mathcal{W}' = (\mathcal{W} \oplus Rl)\text{mod}RL$ for an arbitrary $\mathcal{W} \subseteq \{0, \ldots, RL-1\}$. Then, using the conditions (15), (16), and (25), we see that

$$\begin{aligned}\boldsymbol{\mathcal{H}}_{\mathcal{W}',\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^{m\prime}}[\nu] =& \boldsymbol{\mathcal{H}}_{\mathcal{W},\bullet}\left[\nu + \frac{l}{L}\right]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}\left[\nu + \frac{l}{L}\right]\\ =& \boldsymbol{I}_{\mathcal{W},\mathcal{K}_m} \equiv \boldsymbol{I}_{\mathcal{W}',\mathcal{K}^{m\prime}}\end{aligned}$$

implying that $\boldsymbol{\mathcal{H}}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}[\nu] = \boldsymbol{I}_{\bullet,\mathcal{K}^m}$ for all l and m. Therefore, we obtain $\boldsymbol{\mathcal{H}}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}[\nu] = \boldsymbol{I}_{\bullet,\mathcal{K}}$ for all $\nu \in \mathbb{R}$, where $\mathcal{K} = \bigcup_l \bigcup_m [(\mathcal{K}^m \oplus Rl)\text{mod}RL]$. Equivalently, we have

$$\boldsymbol{\mathcal{H}}^\star[z]\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z] = \boldsymbol{I}_{\bullet,\mathcal{K}} \tag{26}$$

in the Z-domain. If $P < Q$, then $\text{rank}(\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z]) \leq Q-1$, implying that (26) fails to hold. Similarly, if all the $Q \times Q$ minors of $\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z]$ share a common factor of the form $(z - z_0)$ where $z_0 \neq 0$, then $\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z]$ loses rank at $z = z_0$, and this contradicts (26) because $\boldsymbol{\mathcal{H}}^\star[z]$ is FIR and cannot cancel the $(z - z_0)$ factor. This proves the converse statement. □

The import of this result is that for FIR channels, perfect reconstruction is possible using FIR filters, provided that the modulated channel transfer function matrix $\boldsymbol{\mathcal{G}}^\star[z]$ is sufficiently "diverse" in the sense that its null space is empty for all $z \notin \{0, \infty\}$. Of course, we do not care about the cases $z = 0$ or $z = \infty$ because no causality requirement is imposed on the FIR filters. Note, however, that the FIR channel is not a necessary requirement but just a convenient case to consider.

Suppose that $P = Q$; then, $\boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z]$ is a $Q \times Q$ matrix, and the necessary and sufficient condition for perfect reconstruction using FIR filters reduces to

$$\det \boldsymbol{\mathcal{G}}^\star_{\bullet,\mathcal{K}}[z] = Kz^{-d}, \quad K \neq 0,\ d \in \mathbb{Z}.$$

This condition is similar to the perfect reconstruction condition for filterbanks. Even in this special case, however, the result generalizes the filterbanks result by the dependence on the band structure of the inputs via the set $\mathcal{K}$.

We point out some other relations of our results to existing results. The problem in [42] deals with existence of MIMO FIR equalizer filters in the absence of decimation of channel outputs, whereas the classical filter bank problem deals with a single-input multiple-output channel whose outputs are decimated. The present problem deals with the existence of an MIMO FIR reconstruction filter in the presence of decimation. The solution depends not only on the channel transfer function matrix $\boldsymbol{G}[\nu]$ (as in [42]) but on the decimation factor L a well (as in the filter bank problem) and the band-structure of the inputs through $\mathcal{K}$. Thus, Theorem 2 generalizes all these problems simultaneously. In particular, when $L = 1$ and all the discrete-time inputs are full-band signals, our result reduces to that of [42].

Example 2: Let us reconsider the MIMO channel of Example 1. It is easy to check using (23) that $\mathcal{K} = \{0, 1, \ldots, 7\}$. Since $P = 5$ does not satisfy $P \geq Q = |\mathcal{K}| = 8$, by Theorem 2, we cannot achieve perfect reconstruction using FIR filters alone.

V. Minimax Reconstruction Filter Design

A. Reconstruction Error

In this section, we study the problem of reconstruction filter design for a given MIMO sampling scheme. We have seen in Section IV-A that under certain conditions on the channel and the class of input signals, perfect reconstruction is possible. Unfortunately, these ideal filters are not necessarily FIR filters. Conversely, FIR filters do not generally guarantee perfect reconstruction of the channel inputs. Nevertheless, we can approximate the ideal reconstruction filters using FIR filters chosen judiciously so that an appropriate cost function, such as the worst-case end-to-end distortion, is minimized.

We model the input signals as discrete-time multiband functions $X_r[\nu] = 0$, $\nu \notin \mathcal{F}_r$ with $\boldsymbol{x} \in \mathcal{C}$, where $\mathcal{C}$ is the constraint set for the channel inputs:

$$\boldsymbol{x} \in \mathcal{C} = \{\boldsymbol{x} : \|x_r\| \leq \eta_r\}$$

for some $\eta_r > 0$, $r \in \mathcal{R}$, i.e., the input signal energies are upper bounded. The reconstruction filters are approximated by FIR filters, i.e., we enforce the following parameterization on $\boldsymbol{H}[\nu]$:

$$H_{rp}[\nu] = \sum_{k \in \mathcal{Q}_{rp}} h_{rpk} e^{-j2\pi\nu k}, \quad r \in \mathcal{R},\ p \in \mathcal{P} \tag{27}$$

where $\mathcal{Q}_{rp} = \{k : \kappa_{rp} \leq k \leq l_{rp} + \kappa_{rp} - 1\}$ is a finite set representing the locations of the nonzero filter coefficients of $H_{rp}[\nu]$. The quantities l_{rp} and κ_{rp} are the length of the filter and the position of its first nonzero filter coefficient, respectively. Let $\boldsymbol{e}[k] = \tilde{\boldsymbol{x}}[k] - \boldsymbol{x}[k]$ denote the reconstruction error due to the FIR restriction. We will now derive an expression for

$$\boldsymbol{\mathcal{E}}[\nu] = \tilde{\boldsymbol{\mathcal{X}}}[\nu] - \boldsymbol{\mathcal{X}}[\nu], \quad \nu \in \left[0, \frac{1}{L}\right] \tag{28}$$

as a function of the input signals, the channel, and the reconstruction filters alone. Define index sets

$$\mathcal{I}_r = (R\mathcal{L}) \oplus r = \{Rl + r : l \in \mathcal{L}\} \tag{29}$$

$$\mathcal{K}_{r,\nu} = \mathcal{K}_\nu \cap \mathcal{I}_r = \left\{Rl + r : l \in \mathcal{L} \text{ and } \left(\nu + \frac{l}{L}\right) \in \mathcal{F}_r\right\} \tag{30}$$

for each $r \in \mathcal{R}$. It is clear from (9) and (29) that

$$\boldsymbol{\mathcal{X}}_{\mathcal{I}_r}[\nu] = \left(X_r[\nu] \quad X_r\left[\nu + \tfrac{1}{L}\right] \quad \cdots \quad X_r\left[\nu + \tfrac{L-1}{L}\right] \right)^T$$

with a similar expression for $\tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{I}_r}[\nu]$, for each $r \in \mathcal{R}$, i.e., these quantities are the length-L vectorized representations of

$X_r[\nu]$ and $\tilde{X}_r[\nu]$, respectively. Hence, the energy of e_r can be expressed as a function of $\boldsymbol{\mathcal{E}}[\nu]$ using Parseval's theorem:

$$\|e_r\|^2 = \int_{[0,1]} |E_r[\nu]|^2 d\nu = \int_{[0,1/L]} \|\boldsymbol{\mathcal{E}}_{\mathcal{I}_r}[\nu]\|^2 d\nu. \quad (31)$$

Similar relations hold for x_r and other signals in terms of the vectorized version of their discrete-time Fourier transforms. Now, for each $r \in \mathcal{R}$ and $\nu \in [0, 1/L]$, (7) and its analog for $\tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{I}_r}[\nu]$ yield

$$\begin{aligned}\tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{I}_r}[\nu] &= \boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]\boldsymbol{\mathcal{G}}[\nu]\boldsymbol{\mathcal{X}}[\nu] \\ &= \sum_{s\in\mathcal{R}} \boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu]\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu] \end{aligned} \quad (32)$$

where the second step holds because the sets $\{\mathcal{I}_r\}$ partition $\{0, 1, \ldots, RL - 1\}$. Therefore, (28) and (32) give us

$$\begin{aligned}\boldsymbol{\mathcal{E}}_{\mathcal{I}_r}[\nu] &= \tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{I}_r}[\nu] - \boldsymbol{\mathcal{X}}_{\mathcal{I}_r}[\nu] \\ &= \sum_{s\in\mathcal{R}} \boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu]\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu] - \boldsymbol{\mathcal{X}}_{\mathcal{I}_r}[\nu] \\ &= \sum_{s\in\mathcal{R}} \left(\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu] - \delta_{rs}\boldsymbol{I}_L\right)\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu] \end{aligned} \quad (33)$$

where δ_{rs} is the Kronecker delta function, and $\boldsymbol{I}_L$ is the identity matrix of size $L \times L$. Since $\boldsymbol{\mathcal{X}}_{\mathcal{K}_{s,\nu}}[\nu]$ captures all the nonzero components of $\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu]$, we can invoke Proposition 1 to write

$$\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu] = \boldsymbol{E}_{s,\nu}\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu] \quad (34)$$

where

$$\boldsymbol{E}_{s,\nu} \stackrel{\text{def}}{=} \boldsymbol{I}_{\mathcal{I}_s,\mathcal{K}_{s,\nu}}\boldsymbol{I}_{\mathcal{K}_{s,\nu},\mathcal{I}_s} \quad (35)$$

is a diagonal matrix with zeros or ones on the diagonal. Hence, (33) and (34) yield

$$\boldsymbol{\mathcal{E}}_{\mathcal{I}_r}[\nu] = \sum_{s\in\mathcal{R}} \boldsymbol{T}^{rs}[\nu]\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu] \quad (36)$$

$$\boldsymbol{T}^{rs}[\nu] = \left(\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu] - \delta_{rs}\boldsymbol{I}_L\right)\boldsymbol{E}_{s,\nu}. \quad (37)$$

We point out that if $\boldsymbol{H}[\nu]$ is a perfect reconstruction filter matrix, then using (14), it is easily shown that

$$\boldsymbol{T}^{rs}[\nu] = \left(\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu] - \delta_{rs}\boldsymbol{I}_L\right) = \boldsymbol{0}. \quad (38)$$

For simplicity, we rewrite (36) as

$$e_r = \sum_{s\in\mathcal{R}} T^{rs} x_s \quad (39)$$

where $T^{rs} : \mathcal{B}(\mathcal{F}_s) \to l_2$ is the linear operator equivalent of $\boldsymbol{T}^{rs}[\nu]$ acting on x_s.

The norm of the operator T^{rs}, which is needed later, can be computed as follows:

$$\begin{aligned}\|T^{rs}\|^2 &= \sup_{\substack{\|x_s\|\le 1 \\ x_s\in\mathcal{B}(\mathcal{F}_s)}} \|T^{rs}x_s\|^2 \quad (40) \\ &= \sup \int_{\nu\in[0,1/L]} \|\boldsymbol{T}^{rs}[\nu]\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu]\|^2 d\nu \\ &\quad \text{s.t.} \int_\nu \|\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu]\|^2 d\nu \le 1,\ x_s \in \mathcal{B}(\mathcal{F}_s). \quad (41)\end{aligned}$$

Note that the condition $x_s \in \mathcal{B}(\mathcal{F}_s)$ implies that some entries of $\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu]$ necessarily vanish due to (34) because $\boldsymbol{E}_{s,\nu}$ is a diagonal matrix with zeros or ones on the diagonal. In other words, $\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu]$ is not an arbitrary vector in $C^{L\times 1}$. From (35) and (37), it is clear that if the kth component of $\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu]$ vanishes for some k, then the kth column of $\boldsymbol{T}^{rs}[\nu]$ also vanishes. Hence, the range space of $(\boldsymbol{T}^{rs}[\nu])^H$ equals the signal space for input s, namely, $X_s[\nu]$. Now, it follows immediately that we can drop the constraint $x_s \in \mathcal{B}(\mathcal{F}_s)$ in (41) to obtain

$$\|T^{rs}\|^2 = \sup_{\nu\in[0,1/L]} \|\boldsymbol{T}^{rs}[\nu]\|^2 = \sup_{\nu\in[0,1/L]} \sigma_{\max}(\boldsymbol{T}^{rs}[\nu]) \quad (42)$$

where $\|\boldsymbol{T}^{rs}[\nu]\|$ is the spectral norm of matrix $\boldsymbol{T}^{rs}[\nu]$ for each ν, and $\sigma_{\max}(\cdot)$ is the largest singular value.

B. Cost Function

Our goal is to design an FIR reconstruction filter matrix $\boldsymbol{H}[\nu]$ such that a measure of the reconstruction error $\boldsymbol{e}$ is minimized.

From (31) and (36), we have

$$\|e_r\|^2 = \int_{[0,1/L]} \left\| \sum_{s\in\mathcal{R}} \boldsymbol{T}^{rs}[\nu]\boldsymbol{\mathcal{X}}_{\mathcal{I}_s}[\nu], \right\|^2 d\nu. \quad (43)$$

Clearly, the above expression is completely parameterized by the coefficients of the filter $\boldsymbol{H}_{r,\bullet}[\nu]$, namely

$$\boldsymbol{h}^r = \{h_{rpk} :\ p \in \mathcal{P},\ k \in \mathcal{Q}_{rp}\}$$

because $\boldsymbol{T}^{rs}[\nu]$ depends only on $\boldsymbol{H}_{r,\bullet}[\nu]$ and $\boldsymbol{G}_{\bullet,s}[\nu]$. Consequently, for each $r \in \mathcal{R}$, the set of coefficients $\boldsymbol{h}^r$ (or, equivalently, the rth row of $\boldsymbol{H}[\nu]$) can be optimized independently of the others by minimizing a cost that measures the fidelity of reconstruction of the rth input.

We choose the cost function to be the norm of the error e_r in the worst case over all r and $\boldsymbol{x} \in \mathcal{C}$, i.e., $\sup_{x\in\mathcal{C}} \max_{r\in\mathcal{R}} \|e_r\|$. Thus, we seek the solution to

$$\min_{H:\text{FIR}} \sup_{x\in\mathcal{C}} \max_{r\in\mathcal{R}} \|e_r\| = \min_{H:\text{FIR}} \max_{r\in\mathcal{R}} \sup_{x\in\mathcal{C}} \|e_r\|.$$

From our earlier argument, this decouples into the following R independent subproblems:

$$\min_{h^r} \sup_{x\in\mathcal{C}} \|e_r\| = \min_{h^r} C_r(\boldsymbol{h}^r), \quad r \in \mathcal{R}$$

where $C_r(\boldsymbol{h}^r)$ is the cost function associated with the rth output:

$$C_r(\boldsymbol{h}^r) = \sup \|e_r\| \quad \text{s.t.} \quad \|x_s\| \le \eta_s,\ x_s \in \mathcal{B}(\mathcal{F}_s),\ s \in \mathcal{R}. \quad (44)$$

It turns out that (44) is difficult to minimize directly; therefore, we look for an alternate expression for the cost such as a bound on $C_r(\boldsymbol{h}^r)$. The following proposition, which is proved in the Appendix, provides upper and lower bounds on the cost function.

Proposition 3: The cost function $C_r(\boldsymbol{h}^r)$ can be bounded as

$$\frac{1}{\sqrt{R}} \bar{C}_r(\boldsymbol{h}^r) \le C_r(\boldsymbol{h}^r) \le \bar{C}_r(\boldsymbol{h}^r)$$

where

$$\bar{C}_r(\boldsymbol{h}^r) \stackrel{\text{def}}{=} \sum_{s\in\mathcal{R}} \eta_s \|T^{rs}\| = \sum_{s\in\mathcal{R}} \eta_s \sup_{\nu\in[0,1/L]} \|\boldsymbol{T}^{rs}[\nu]\|.$$

Instead of minimizing $C_r(\boldsymbol{h}^r)$ to compute the optimal filter coefficients $\boldsymbol{h}^r$, we minimize $\bar{C}_r(\boldsymbol{h}^r)$ as it produces a considerably simpler algorithm to implement. Therefore, the approximate optimal filter coefficients are given by

$$\boldsymbol{h}_\circ^r = \arg\min_{\boldsymbol{h}^r} \bar{C}_r(\boldsymbol{h}^r), \quad \bar{C}_r(\boldsymbol{h}^r) = \sum_{s\in\mathcal{R}} \eta_s \sup_{\nu\in[0,1/L]} \|\boldsymbol{T}^{rs}[\nu]\|. \tag{45}$$

Owing to Proposition 3 , the approximate solution will produce a cost that is greater by a factor of not more than $\sqrt{R}$ times the true minimum, i.e.,

$$\min_{\boldsymbol{h}^r} C_r(\boldsymbol{h}^r) \le \bar{C}_r(\boldsymbol{h}_\circ^r) \le \sqrt{R}\min_{\boldsymbol{h}^r} C_r(\boldsymbol{h}^r). \tag{46}$$

C. Asymptotics of the Approximation Error

An important question pertaining to the FIR design is whether the resulting approximation error goes to zero when the filter lengths go to infinity. Under some conditions, we can answer affirmatively, as the following theorem shows.

Theorem 3: Suppose that $\boldsymbol{G}[\nu]$ is continuous and that $\boldsymbol{H}[\nu]$ has an FIR parameterization described in (27). If there exists a perfect reconstruction filter matrix continuous in ν, then $\min_{\boldsymbol{h}^r} C_r(\boldsymbol{h}^r) \to 0$ as $\kappa_{rp} \to -\infty$ and $\kappa_{rp} + l_{rp} \to \infty$.

Theorem 3, which is proved in the Appendix, guarantees the existence of continuous FIR solutions $\boldsymbol{H}[\nu]$ that can get arbitrarily close to perfect reconstruction, provided that there exists a continuous $\boldsymbol{H}^\circ[\nu]$ with the perfect reconstruction property. Furthermore, empirical evidence suggests that for the optimal FIR approximation, the worst-case reconstruction error falls off exponentially fast with increasing filter length. Combining this fact with (46) (which follows from Proposition 3) suggests that the impact of using the surrogate cost function $\bar{C}(\boldsymbol{h})$ instead of $C(\boldsymbol{h})$ has a minimal impact: At most, a usually negligible $O(\log R)$ increase in filter length is required to meet a fixed error tolerance.

D. Semi-Infinite Linear Program Formulation

Next, we present an algorithm to compute the optimal solution to the problem in (45). We show that this problem can be reduced to a *semi-infinite linear program*, which can then be solved by a standard method.

We begin by expressing the matrices $\boldsymbol{T}^{rs}[\nu]$ as functions of the filter coefficients $\boldsymbol{h}^r$.

Proposition 4: The quantity $\boldsymbol{T}^{rs}[\nu]$ defined in (37) can be written as

$$\boldsymbol{T}^{rs}[\nu] = \boldsymbol{F}_0^{rs}[\nu] + \sum_{p\in\mathcal{P}}\sum_{k\in\mathcal{Q}_{rp}} h_{rpk}\boldsymbol{F}_{pk}^{rs}[\nu] \tag{47}$$

for appropriate matrices $\boldsymbol{F}_0^{rs}[\nu]$ and $\boldsymbol{F}_{pk}^{rs}[\nu]$.

Proof: Observe from (11) and (27) that the (l,p) entry of $\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]$ is given by

$$[\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]]_{lp} = H_{rp}\left[\nu + \frac{l}{L}\right] = \sum_{k\in\mathcal{Q}_{rp}} h_{rpk}e^{-j2\pi(\nu+l/L)k}.$$

In other words, $\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu]$ can be written as the following linear combination:

$$\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu] = \sum_{p\in\mathcal{P}}\sum_{k\in\mathcal{Q}_{rp}} h_{rpk}\boldsymbol{K}^{rpk}[\nu] \tag{48}$$

where $\boldsymbol{K}^{rpk}[\nu]$ are matrices whose entries are

$$[\boldsymbol{K}^{rpk}[\nu]]_{lp'} = \delta_{pp'}e^{-j2\pi(\nu+l/L)k}.$$

Combining (37) and (48), we obtain the desired affine form in (47), where

$$\boldsymbol{F}_0^{rs}[\nu] = -\delta_{rs}\boldsymbol{E}_{s,\nu} \quad \text{and} \quad \boldsymbol{F}_{pk}^{rs}[\nu] = \boldsymbol{K}^{rpk}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu]\boldsymbol{E}_{s,\nu}$$

are the explicit expressions for the matrices involved, provided for the sake of completeness.

Proposition 4 shows that $\boldsymbol{T}^{rs}[\nu]$ has an affine form in terms of the filter coefficients $\boldsymbol{h}^r$. Next, recall that

$$\|\boldsymbol{T}^{rs}[\nu]\| = \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{O}} \Re\Big(\boldsymbol{y}^H\boldsymbol{T}^{rs}[\nu]\boldsymbol{x}\Big)$$

where $\|\boldsymbol{T}\|$ denotes the spectral norm of a matrix $\boldsymbol{T}$. Therefore, for a fixed index r, we rewrite the optimization in (45) as

$$\min \sum_{s\in\mathcal{R}} \eta_s\delta_s \quad \text{s.t.} \quad \delta_s \ge \Re\Big(\boldsymbol{y}^H\boldsymbol{T}^{rs}[\nu]\boldsymbol{x}\Big) \quad \forall s\in\mathcal{R},\ \forall\nu\in\left[0,\ \frac{1}{L}\right],\ \text{and } \forall\boldsymbol{x},\ \boldsymbol{y}\in\mathcal{O} \tag{49}$$

where $\Re(\cdot)$ denotes the real part, and $\mathcal{O} = \big\{\boldsymbol{v}\in\mathbb{C}^L : \|\boldsymbol{v}\| \le 1\big\}$ is the unit ball for length-L vectors. For convenience, we treat $\boldsymbol{h}^r$ as a row-vector (with any ordering of coefficients), i.e.,

$$\boldsymbol{h}_{j(p,k)}^r = h_{rpk}$$

where $j(p,k)$ is an invertible mapping that takes the pair of indices $p\in\mathcal{P}$ and $k\in\mathcal{Q}_{rp}$ to a single index j in the set $\mathcal{J}_r$ defined as

$$\mathcal{J}_r = \{0,\dots,J_r-1\},\quad J_r \stackrel{\text{def}}{=} \sum_{p\in\mathcal{P}}|\mathcal{Q}_{rp}| = \sum_{p\in\mathcal{P}} l_{rp}.$$

Recall that $\mathcal{Q}_{rp} = \big\{k : \kappa_{rp} \le k \le l_{rp}+\kappa_{rp}-1\big\}$. Hence, an example of one such mapping is

$$j(p,k) = \sum_{p'=0}^{p-1} l_{rp'} + (k-\kappa_{rp}).$$

Define a row-vector $\boldsymbol{\delta} = \big[\delta_0 \ \cdots \ \delta_{R-1}\big]$. Using the affine representation in (47), we can rewrite (49) as

$$\min\sum_{s\in\mathcal{R}}\eta_s\delta_s \text{ s.t.} \quad \Re\left(\delta_s - \sum_{p\in\mathcal{P}}\sum_{k\in\mathcal{Q}_{rp}} h_{rpk}\big(\boldsymbol{y}^H\boldsymbol{F}_{pk}^{rs}[\nu]\boldsymbol{x}\big)\right) \ge \Re\big(\boldsymbol{y}^H\boldsymbol{F}_0^{rs}[\nu]\boldsymbol{x}\big), \forall(s,\nu,\boldsymbol{x},\boldsymbol{y})\in\mathcal{U}$$

where $\mathcal{U} = \mathcal{R}\times[0,1/L]\times\mathcal{O}\times\mathcal{O}$. This problem can be recast as

$$\min\Re(\boldsymbol{c\xi}) \quad \text{s.t.} \quad \Re\big(\boldsymbol{a}(\boldsymbol{u})\boldsymbol{\xi}\big) \ge \Re\big(b(\boldsymbol{u})\big),\ \ \forall\boldsymbol{u}\in\mathcal{U} \tag{50}$$

where $\boldsymbol{\xi} = [\boldsymbol{\delta} \quad \boldsymbol{h}^r]$ is the set of program variables, $\boldsymbol{u} = (s, \nu, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{U}$ parameterizes the constraints, and $b(\boldsymbol{u})$ is a complex-scalar. The problem (50) is a *semi-infinite linear program* because the number of program variables is finite while the number constraints (cardinality of $\mathcal{U}$) is infinite. The quantities $\boldsymbol{a}(\boldsymbol{u})$ and $\boldsymbol{c}$ are row-vectors of length $R + J_r$, whose first R entries are real, and the remaining J_r entries are complex-valued:

$$\boldsymbol{a}_n(\boldsymbol{u}) = \begin{cases} 1, & \text{if } n = s \\ 0, & \text{if } n \in \mathcal{R},\ n \neq s \\ -\boldsymbol{y}^H \boldsymbol{F}_{pk}^{rs}[\nu]\boldsymbol{x}, & \text{if } n = R + j(p,k) \end{cases}$$
$$b(\boldsymbol{u}) = \boldsymbol{y}^H \boldsymbol{F}_0^{rs}[\nu]\boldsymbol{x}$$
$$\boldsymbol{c}_n = \begin{cases} \eta_s, & \text{if } n = \mathcal{R} \\ 0, & \text{otherwise.} \end{cases}$$

The semi-infinite program in (50) is in a nonstandard form since it contains a mixture of real and complex variables. Nevertheless, it can be converted to the standard real form by decomposing all complex variables into their real and imaginary parts. Finding the dual of this real program and reconverting to the complex form produces the following dual program:

$$\max \int_{\mathcal{U}} b(\boldsymbol{u}) dw(\boldsymbol{u}) \quad \text{s.t.} \quad \boldsymbol{c} = \int_{\mathcal{U}} \boldsymbol{a}(\boldsymbol{u}) dw(\boldsymbol{u}), \quad w \geq 0 \tag{51}$$

where w is a real and positive measure on $\mathcal{U}$. The optimal solution w has a point-distribution with cardinality no more than $2J_r + R$, i.e.,

$$\max \sum_{j=1}^{2J_r+R} b(\boldsymbol{u}_j) w_j \quad \text{s.t.} \quad \boldsymbol{c} = \sum_{j=1}^{2J_r+R} \boldsymbol{a}(\boldsymbol{u}_j) w_j, \quad w \geq 0 \tag{52}$$

where $w_j \geq 0$ is the intensity of the point-distribution at $\boldsymbol{u}_j \in \mathcal{U}$. The solution to (52) can be found using a simplex-type algorithm for semi-infinite programs [35]. The method involves pivoting starting from an intitial feasible solution. Since there is no duality gap, the primal and dual solutions satisfy

$$\boldsymbol{a}(\boldsymbol{u}_j)\boldsymbol{\xi} = b(\boldsymbol{u}_j), \quad j = 1, \ldots 2J_r + R. \tag{53}$$

Once the optimal dual solution $(\boldsymbol{u}_j, w_j)$ has been found, the primal solution can be computed by solving (53) for $\boldsymbol{\xi}$. At any stage of the dual algorithm, the dual cost is a lower bound on the optimal cost. An upper bound on the optimal cost can be computed by solving (53) with the program variables $\boldsymbol{u}_j$ during any stage of the dual algorithm. Since the pivoting does not terminate in a finite number of steps, we stop when the lower bound (dual cost) is close enough to the upper bound. The analysis of the computational complexity of this algorithm is beyond the scope of this paper.

In summary, the semi-infinite linear program (50) and its dual (51) are expressed in terms of $\boldsymbol{a}(\boldsymbol{u})$, $b(\boldsymbol{u})$, and $\boldsymbol{c}$, which ultimately depend on the channel $\boldsymbol{G}(f)$, the band-structures of the channel inputs, and the weights η_s, $s \in \mathcal{R}$. The dual problem is solved using a simplex-type algorithm.

Recall that whenever the technical conditions in Theorem 1 are satisfied, the set

$$\mathcal{S}_H = \{\boldsymbol{H}[\nu] : \text{perfect reconstruction is achieved.}\}$$

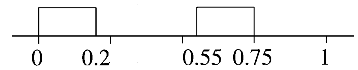

Fig. 5. Indicator function of the spectral support $\mathcal{F}$ for Example 3.

is nonempty. However, $\mathcal{S}_H$ need not be a singleton set because the perfect reconstruction filter matrices are not necessarily unique. The optimization always produces the FIR filter matrix that is closest to the set of reconstruction filter matrices $\mathcal{S}_H$ in the sense that it minimizes the cost function that represents the worst-case approximation error energy over all inputs $\boldsymbol{x} \in \mathcal{C}$ due to imperfect reconstruction. If the conditions for continuity in Theorem 1 are satisfied, then $\mathcal{S}_H$ contains a continuous $\boldsymbol{H}[\nu]$ and guarantees, by Theorem 3, that the approximation error can be made arbitrarily small by using sufficiently long FIR filters.

E. Design Examples

In this section, we consider two FIR filter design examples. In the first example, we design reconstruction filters for the multicoset sampling scheme that is a special case of MIMO sampling [29], [30]. In the second example, we consider MIMO sampling using a channel having two inputs and five outputs. The semi-infinite algorithm was programmed using MATLAB and C.

Example 3: In this example, we design FIR reconstruction filters for multicoset sampling, which is a scheme where a single (scalar) multiband signal is sampled on a nonuniform but periodic set of locations.

Let $\mathcal{F} = [0, 0.2) \cup [0.55, 0.75)$, as illustrated in Fig. 5, be the spectral support for the class of signals to be subsampled. The Landau lower bound on the downsampling rate for this spectral support is 0.4 (the total measure of $\mathcal{F}$). However, the minimum downsampling rate that can be achieved for this spectral support by uniform sampling is only 0.75 because translates of $\mathcal{F}$ do not pack efficiently. Instead, consider nonuniform subsampling on the set

$$\Lambda = \bigcup_{n \in \mathbb{Z}} \{4n, 4n+1\}$$

for signals in $\mathcal{B}(\mathcal{F})$. This corresponds to nonuniform downsampling by a factor of two, or a downsampling rate of 0.5, which is just slightly higher than the Landau rate, and a factor of 1.5 improvement over the best uniform subsampling rate. The sampling set Λ is clearly a union of two uniformly subsampled streams, namely, $\{4n : n \in \mathbb{Z}\}$ and $\{4n+1 : n \in \mathbb{Z}\}$. Therefore, this sampling scheme can be recast as the uniform MIMO sampling (see Fig. 2) with one input, two outputs, and $L = 4$. The first channel output is the input itself, and the second output is the input delayed by one sample so that the subsampled outputs produce the two desired input streams. Thus, we have

$$\boldsymbol{G}[\nu] = \begin{pmatrix} 1 \\ e^{-j2\pi\nu} \end{pmatrix}.$$

For this single-input double-output channel, we seek the optimal 2×1 FIR reconstruction filter matrix $\boldsymbol{H}[\nu] = [H_0[\nu] \quad H_1[\nu]]$, where each of the filters $H_p[\nu]$ is an FIR filter with impulse response of length 21 centered at the origin, i.e., $\mathcal{Q}_{1p} = \{-10, \ldots, 10\}$, $p = 0, 1$. Since $R = 1$, we can take $\eta_0 = 1$ without loss of generality. Applying

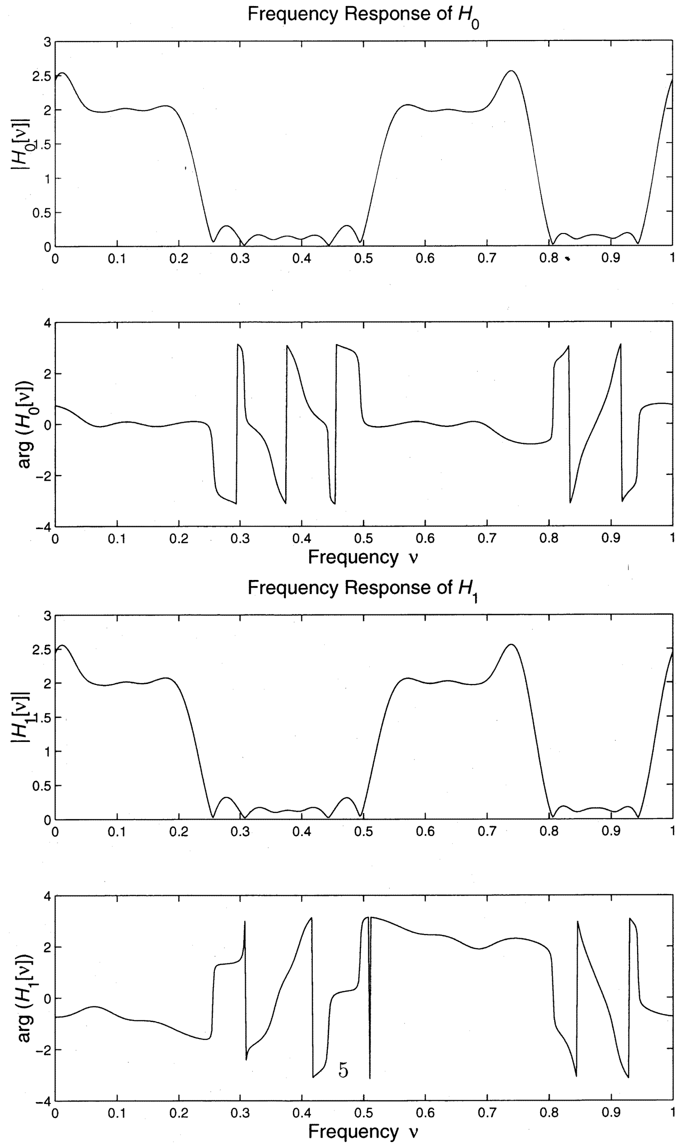

Fig. 6. Magnitude and phase responses of the optimal FIR filters $H_0[\nu]$ and $H_1[\nu]$.

the semi-infinite algorithm, we obtain the optimal FIR filters $H_0[\nu]$ and $H_1[\nu]$, which are shown in Fig. 6. The resulting maximum approximation error $\|\boldsymbol{T}^{00}[\nu]\|$ at optimality is shown for $\nu \in [0, 1/4)$ in Fig. 7. The equal-ripple nature of this plot is due to the minimax criterion:

$$\delta = \min_{\boldsymbol{h}^1} C_0(\boldsymbol{h}^0) = \min_{\boldsymbol{h}^0} \sup_{\nu} \|\boldsymbol{T}^{00}[\nu]\|.$$

The optimal cost is $\delta = 0.1348$.

Example 4: Consider the 2×5 MIMO system with inputs $x_0[k]$ and $x_1[k]$ described in Example 1. We have already seen that the existence of a continuous perfect reconstruction filter matrix $\boldsymbol{H}[\nu]$ is guaranteed. As a consequence of Theorem 3, the approximation error approaches zero as the filter lengths are increased.

Let $\eta_0 = \eta_1 = 0.5$ be the bounds on the two-norms of the inputs. Using the semi-infinite algorithm, we design six sets of

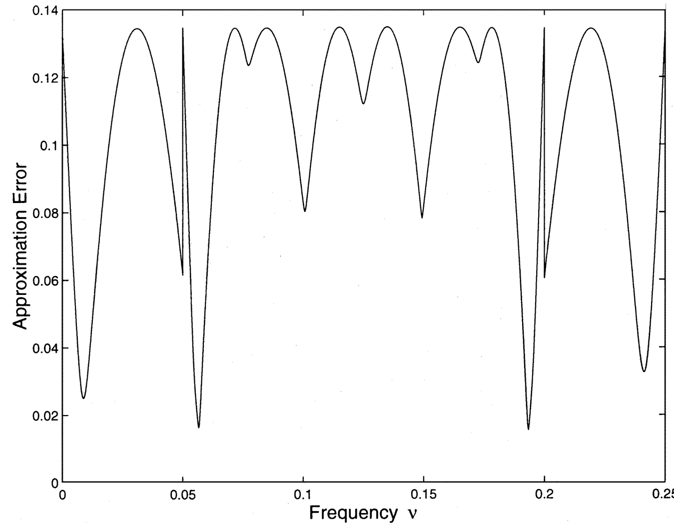

Fig. 7. Approximation error $\|\boldsymbol{T}^{00}[\nu]\|$ at optimality for Example 3.

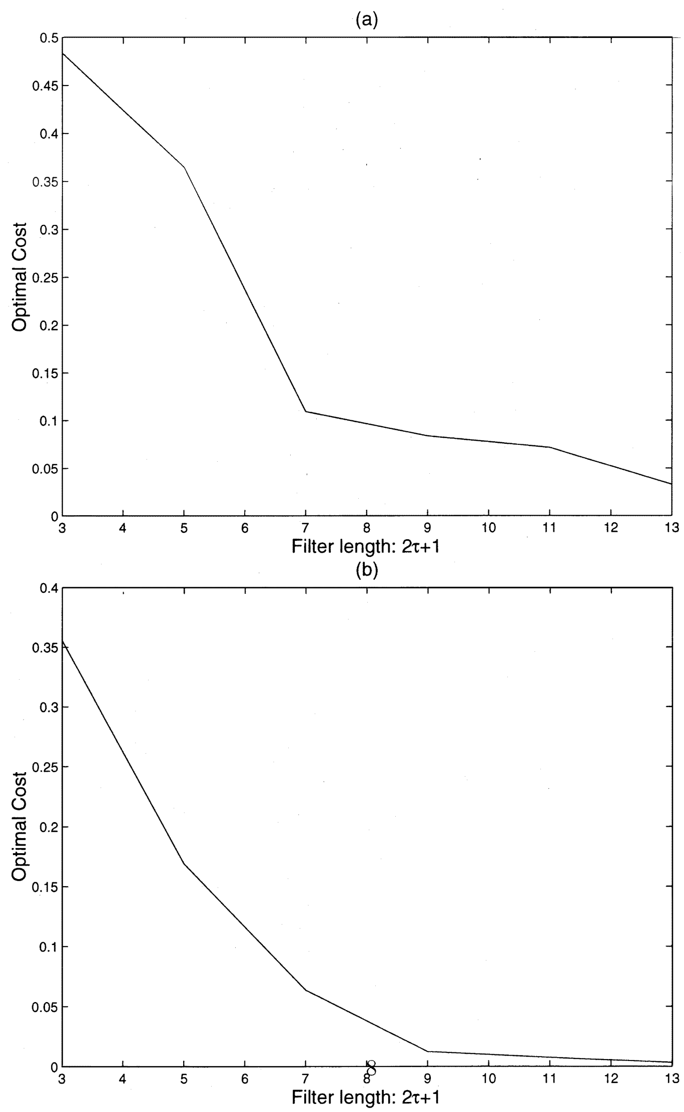

Fig. 8. Optimal costs (a) $\bar{C}_0(\boldsymbol{h}_\circ^0)$ and (b) $\bar{C}_1(\boldsymbol{h}_\circ^1)$ for FIR reconstruction filters of length $2\tau + 1$, $1 \leq \tau \leq 6$.

TABLE I
COST FUNCTIONS $\bar{C}_0(\boldsymbol{h}_\circ^0)$ AND $\bar{C}_1(\boldsymbol{h}_\circ^1)$ AT OPTIMALITY FOR FIR RECONSTRUCTION FILTERS OF LENGTH $2\tau+1, 1 \le \tau \le 6$

$2\tau+1$	3	5	7	9	11	13
$\bar{C}_0(\boldsymbol{h}_\circ^0)$	0.4835	0.3643	0.1093	0.0836	0.0716	0.0329
$\bar{C}_1(\boldsymbol{h}_\circ^1)$	0.3554	0.1690	0.0637	0.0124	0.0076	0.0034

reconstruction filters of varying filter lengths, indexed by $\tau \in \{1, 2, \ldots, 6\}$, having the following specifications:

$$l_{rp} = 2\tau + 1$$
$$\kappa_{rp} = \begin{cases} 0 - \tau, & \text{if } p \in \{0, 1\} \\ 1 - \tau, & \text{if } p \in \{2,\ 3,\ 4\}. \end{cases}$$

In other words, all the FIR reconstruction filters for a given τ have equal lengths ($2\tau + 1$). Furthermore, the filters $h_{pr}[k]$ are centered at $k = 0$ for $p = 0, 1$ and at $k = 1$ for $p = 2, 3, 4$. Table I and Fig. 8 show the cost functions for the two outputs and the six design cases. Observe that the cost falls off quickly as the filter lengths increase.

Therefore, as discussed earlier, the use of the surrogate cost function $\bar{C}(\boldsymbol{h})$ instead of $C(\boldsymbol{h})$ leads to at most a slight increase in the lengths of filter required to meet a fixed error tolerance. In this example, $\sqrt{R} = \sqrt{2}$ so that by (46), the required increase in length is two or less in most cases listed in Table I. Finally, in this example, the costs would converge to zero as $\tau \to \infty$ since the conditions required in Theorem 1 are satisfied.

VI. CONCLUSION

We examined the problem of FIR reconstruction filter design for uniform MIMO sampling of multiband signals with different band structures. The analysis is facilitated by the conversion to an equivalent hypothetical discrete-time system. We presented necessary and sufficient conditions for perfect reconstruction of the channel inputs with and without a continuity requirement on the transfer functions of the reconstruction filters. We also presented necessary and sufficient conditions for the existence of FIR perfect reconstruction filters when the channel itself is FIR. These conditions, which depend on the channel, input multiband structures, and downsampling rate, generalize previous results on multichannel deconvolution and filterbanks. In general, perfect reconstruction FIR filters do not exist for the MIMO sampling problem. Therefore, from an implementation viewpoint, we considered the problem of FIR approximation to the reconstruction system. The continuity property was shown to be important in this context, as it allows us to make the signal reconstruction error arbitrarily small by designing sufficiently long filters in the FIR approximation.

Finally, we formulated the reconstruction filter design problem as a minimax optimization, which was recast as a standard semi-infinite linear program and solved efficiently by computer. The generality of the MIMO setting allows this algorithm to be used for various other sampling schemes that fit into the MIMO framework as special cases.

APPENDIX

Proof of Proposition 3

From (39) and (44), we obtain an upper bound on the cost function

$$\begin{aligned} C_r(\boldsymbol{h}^r) &= \sup \Big\| \sum_{s\in\mathcal{R}} T^{rs} x_s \Big\| \\ \text{s.t.} \quad \|x_s\| &\le \eta_s,\ x_s \in \mathcal{B}(\mathcal{F}_s),\ s \in \mathcal{R} \\ &\le \sup \sum_{s\in\mathcal{R}} \big\| T^{rs} x_s \big\| \\ \text{s.t.} \quad \|x_s\| &\le \eta_s,\ x_s \in \mathcal{B}(\mathcal{F}_s),\ s \in \mathcal{R} \\ &= \sum_{s\in\mathcal{R}} \eta_s \|T^{rs}\| = \sum_{s\in\mathcal{R}} \eta_s \sup_{\nu\in[0,1/L]} \|\boldsymbol{T}^{rs}[\nu]\| \end{aligned}$$

where the last step follows from (42). Hence, $C_r(\boldsymbol{h}^r) \le \bar{C}_r(\boldsymbol{h}^r)$.

To prove the other inequality, we start by choosing a set of signals $x'_s \in \mathcal{B}(\mathcal{F}_s)$, $s \in \mathcal{R}$ such that $\|x'_s\| = \eta_s$ and $\|T^{rs} x'_s\| = \eta_s \|T^{rs}\| - \epsilon/R$, where $\epsilon > 0$. In view of (40), such x'_s exist for any $\epsilon > 0$. First, let $\bar{x}_0 = x'_0$. Then, for $s = 1, 2, \ldots, R-1$, let $\bar{x}_s$ be either x'_s or $-x'_s$ such that

$$\Big\langle T^{rs}\bar{x}_s, \sum_{\sigma=0}^{s-1} T^{r\sigma}\bar{x}_\sigma \Big\rangle \ge 0. \tag{A.1}$$

Now, (A.1) implies that for any $s \in \mathcal{R}$

$$\begin{aligned} \Big\| \sum_{\sigma=0}^{s} T^{r\sigma}\bar{x}_\sigma \Big\|^2 &= \|T^{rs}\bar{x}_s\|^2 + 2\Big\langle T^{rs}\bar{x}_s, \sum_{\sigma=0}^{s-1} T^{r\sigma}\bar{x}_\sigma \Big\rangle \\ &\quad + \Big\| \sum_{\sigma=0}^{s-1} T^{r\sigma}\bar{x}_\sigma \Big\|^2 \\ &\ge \|T^{rs}\bar{x}_s\|^2 + \Big\| \sum_{\sigma=0}^{s-1} T^{r\sigma}\bar{x}_\sigma \Big\|^2. \end{aligned} \tag{A.2}$$

Therefore

$$\begin{aligned} C_r(\boldsymbol{h}^r)^2 &\overset{(a)}{\ge} \Big\| \sum_{\sigma=0}^{R-1} T^{r\sigma}\bar{x}_\sigma \Big\|^2 \overset{(b)}{\ge} \sum_{s\in\mathcal{R}} \|T^{rs}\bar{x}_s\|^2 \\ &\overset{(c)}{=} \sum_{s\in\mathcal{R}} \eta_s^2 \|T^{rs}\|^2 - \epsilon \end{aligned} \tag{A.3}$$

where (a) follows from the definition of $C_r(\boldsymbol{h}^r)$, (b) by recursively applying (A.2) starting with $s = R-1$, and (c) by the choice of $\bar{x}_s$. Now, using the Cauchy-Schwarz inequality, we have

$$\sum_{s\in\mathcal{R}} \eta_s^2 \|T^{rs}\|^2 \ge \frac{1}{R}\Big(\sum_{s\in\mathcal{R}} \eta_s \|T^{rs}\| \Big)^2 = \frac{1}{R}\bar{C}_r(\boldsymbol{h}^r)^2. \tag{A.4}$$

Finally, from (A.3) and (A.4) and because the x'_s can be chosen to make ϵ arbitrarily small, we obtain the other desired inequality: $C_r(\boldsymbol{h}^r) \ge \bar{C}_r(\boldsymbol{h}^r)/\sqrt{R}$.

Proof of Theorem 3

In view of Proposition 3, it suffices to prove that $\lim_{\tau\to\infty} \min_{\boldsymbol{h}^r} \|T^{rs}\| \to 0$ for all $r, s \in \mathcal{R}$, where

$$\tau = \min\big(\{-\kappa_{rp} : r \in \mathcal{R},\ p \in \mathcal{P}\} \cup \{\kappa_{rp} + l_{rp} : r \in \mathcal{R},\ p \in \mathcal{P}\} \big).$$

Suppose that $\boldsymbol{H}^\circ[\nu]$ is continuous in ν and achieves perfect reconstruction. In addition, let $\boldsymbol{\mathcal{H}}^\circ[\nu]$ be the modulated reconstruction matrix corresponding to $\boldsymbol{H}^\circ[\nu]$. From (38), we conclude that

$$\left(\boldsymbol{\mathcal{H}}^\circ_{\mathcal{I}_r,\bullet}[\nu]\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu] - \delta_{rs}\boldsymbol{I}_L\right)\boldsymbol{E}_{s,\nu} = \boldsymbol{0} \tag{A.5}$$

for all $r, s \in \mathcal{R}$ because we would be guaranteed perfect reconstruction if we chose $\boldsymbol{\mathcal{H}}[\nu] = \boldsymbol{\mathcal{H}}^\circ[\nu]$. Then, combining (37) and (A.5), we obtain

$$\boldsymbol{T}^{rs}[\nu] = \left(\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu] - \boldsymbol{\mathcal{H}}^\circ_{\mathcal{I}_r,\bullet}[\nu]\right)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu]\boldsymbol{E}_{s,\nu}$$

for any reconstruction matrix $\boldsymbol{\mathcal{H}}[\nu]$. Therefore

$$\begin{aligned}&\sup_{\nu\in[0,1/L]}\left\|\boldsymbol{T}^{rs}[\nu]\right\| \\ &\le \sup_{\nu\in[0,1/L]}\left\|\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu] - \boldsymbol{\mathcal{H}}^\circ_{\mathcal{I}_r,\bullet}[\nu]\right\| \cdot \left\|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu]\boldsymbol{E}_{s,\nu}\right\| \\ &\le C\sup_{\nu\in[0,1/L]}\left\|\boldsymbol{\mathcal{H}}_{\mathcal{I}_r,\bullet}[\nu] - \boldsymbol{\mathcal{H}}^\circ_{\mathcal{I}_r,\bullet}[\nu]\right\|\cdot\end{aligned}$$

where

$$C = \sup_{\nu\in[0,1/L]}\left\|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{I}_s}[\nu]\boldsymbol{E}_{s,\nu}\right\|$$

is finite because $\boldsymbol{\mathcal{G}}[\nu]$ is a continuous function on the compact set $[0, 1/L]$, and $\boldsymbol{E}_{s,\nu}$ is a constant on each $\mathcal{I}_m$. Using (11) and (29), we obtain

$$\begin{aligned}\sup_{\nu\in[0,1/L]}\left\|\boldsymbol{T}^{rs}[\nu]\right\| \le& C\sqrt{L}\sup_{\nu\in[0,\,1]}\|\boldsymbol{H}_{r,\bullet}[\nu] - \boldsymbol{H}^\circ_{r,\bullet}[\nu]\| \\ \le& C\sqrt{LP}\max_p\sup_{\nu\in[0,\,1]}\|H_{rp}[\nu] - H^\circ_{rp}[\nu]\|.\end{aligned} \tag{A.6}$$

Now, suppose that $\boldsymbol{H}[\nu]$ has an FIR parameterization as in (27), i.e.,

$$H_{rp}[\nu] = \sum_{k\in\mathcal{Q}_{rp}} h_{rpk}e^{-j2\pi\nu k}$$

where $\mathcal{Q}_{rp} = \{\kappa_{rp}, \ldots, \kappa_{rp} + l_{rp}\}$. Then, clearly, each entry of $\boldsymbol{H}[\nu]$ can be expressed as a trigonometric polynomial of degree at least τ. Moreover, the coefficients of the polynomial can be individually controlled by changing the parameters $\boldsymbol{h}^r$. Equivalently, we can reparameterize so that the new parameters are the coefficients of the trigonometric polynomials (rather than the filter coefficients). Now, by the Stone–Weierstrass theorem [43], we obtain

$$\min_{\boldsymbol{h}^r}\sup_{\nu\in[0,\,1]}\|H_{rp}[\nu] - H^\circ_{rp}[\nu]\| \le \epsilon \tag{A.7}$$

for any $\epsilon > 0$ if τ is sufficiently large. Combining (A.6) and (A.7), we obtain the desired result

$$\lim_{\tau\to\infty}\min_{\boldsymbol{h}^r}\sup_{\nu\in[0,1/L]}\left\|\boldsymbol{T}^{rs}[\nu]\right\| = 0.$$

Incidentally, this also proves that $\lim_{\tau\to\infty}\min_{\boldsymbol{h}^r}\bar{C}_r(\boldsymbol{h}^r) = 0$ by Proposition 3

References

[1] B. R. Petersen and D. D. Falconer, "Supression of adjacent-channel, co-channel, and intersimbol interference by equalizers and linear combiners," *IEEE Trans. Commun.*, vol. 42, pp. 3109–3118, Dec. 1994.

[2] J. Yang and S. Roy, "On joint receiver and transmitted optimization for multiple-input-multiple-output (MIMO) transmission systems," *IEEE Trans. Commun.*, vol. 42, pp. 3221–3231, Dec. 1994.

[3] G. G. Raleigh and J. M. Cioffi, "Spatio-temporal coding for wireless communication," *IEEE Trans. Commun.*, vol. 46, pp. 357–366, Mar. 1998.

[4] L. Ye and K. J. R. Liu, "Adaptive blind source separation and equalization for multiple-input/multiple-output systems," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2864–2876, Nov. 1998.

[5] W. Zishun and J. D. Z. chen, "Blind separation of slow waves and spikes from gastrointestinal myoelectrical recordings," *IEEE Trans. Inform. Technol. Biomed.*, vol. 5, pp. 133–137, June 2001.

[6] V. Zarzoso and A. K. Nandi, "Noninvasive fetal electrocardiogram extraction: Blind separation versus adaptive noise cancellation," *IEEE Trans. Biomed. Eng.*, vol. 48, pp. 12–18, Jan. 2001.

[7] P. a. Voois and J. M. Cioffi, "Multichannel signal processing for multiple-head digital magnetic recording," *IEEE Trans. Magn.*, vol. 30, pp. 5100–5114, Nov. 1994.

[8] K.-C. Yen and Y. Zhao, "Adaptive co-channel speech separation and recognition," *IEEE Trans. Speech Audio Process.*, vol. 7, pp. 138–151, Mar. 1999.

[9] A. González and J. J. Lopéz, "Fast transversal filters for deconvolution in multichannel sound reproduction," *IEEE Trans. Speech Audio Processing*, vol. 9, pp. 429–440, May 2001.

[10] J. Idier and Y. Goussard, "Multichannel seismic deconvolution," *IEEE Trans. Geosci Remote Sensing*, vol. 31, no. 5, pp. 961–979, Sept. 1993.

[11] G. Harikumar and Y. Bresler, "Exact image deconvolution from multiple FIR blurs," *IEEE Trans. Image Processing*, vol. 8, pp. 846–862, June 1999.

[12] G. B. Giannakis and R. W. Heath, Jr., "Blind identification of multichannel FIR blurs and perfect image restoration," *IEEE Trans. Image Processing*, vol. 9, pp. 1877–1896, Nov. 2000.

[13] R. Venkataramani and Y. Bresler, "Sampling theorems for uniform and periodic nonuniform MIMO sampling of multiband signals," *IEEE Trans. Signal Processing*, vol. 51, pp. 3152–3163, Dec. 2003.

[14] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[15] R. Venkataramani and Y. Bresler, "Multiple-input multiple-output sampling: Necessary density conditions and stability issues," *IEEE Trans. Inform. Theory*, to be published.

[16] R. Venkataramani, "Sub-Nyquist multicoset and MIMO sampling: Perfect reconstruction, performance analysis, and necessary density conditions," Ph.D. dissertation, Univ. Illinois, Urbana-Champaign, IL, 2001.

[17] R. J. Papoulis, "Generalized sampling expansions," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 652–654, Nov. 1977.

[18] K. Cheung and R. Marks, "Image sampling below the nyquist density without aliasing," *J. Opt. Soc. Amer. A*, vol. 7, no. 1, pp. 92–105, Jan. 1990.

[19] R. E. Kahn and B. Liu, "Sampling representations and the optimum reconstruction of signals," *IEEE Trans. Inform. Theory*, vol. IT-11, pp. 339–347, July 1965.

[20] P. Feng and Y. Bresler, "Spectrum-blind minimum-rate sampling and reconstruction of multi-band signals," *Proc. IEEE Int. Conf. Acoust. Speech, Signal Processing*, May 1996.

[21] C. Herley and P. W. Wong, "Minimum rate sampling of signals with arbitrary frequency support," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1555–6, July 1999.

[22] P. Vaidyanathan and V. Liu, "Efficient reconstruction of band-limited sequences from nonuniformly decimated versions by use of polyphase filter banks," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1927–1936, Nov. 1990.

[23] J. L. Brown, Jr., "Sampling expansions for multi-band signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP–33, pp. 312–315, Feb. 1985.

[24] S. C. Scoluar and W. J. Fitzgerald, "Periodic nonuniform sampling of multi-band signals," *Signal Process.*, vol. 28, no. 2, pp. 195–200, Aug. 1992.

[25] R. G. Shenoy, "Nonuniform sampling of signals and applications," in *Proc. Int. Symp. Circuits Syst.*, vol. 2, London, U.K., May 1994, pp. 181–184.

[26] K. Cheung, “A multidimensional extension of papoulis’ generalized sampling expansion with the application in minimum density sampling,” in *Advanced Topics in Shannon Sampling and Interpolation Theory*, R. J. Marks, II, Ed. New York: Springer-Verlag, 1993, pp. 85–119.

[27] B. Foster and C. Herley, “Exact reconstruction from periodic nonuniform sampling of signals with arbitrary frequency support ,” *Proc. IEEE Int. Conf. Acoust. Speech Signal Processing*, May 1998.

[28] Y. Bresler and P. Feng, “Spectrum-blind minimum-rate sampling and reconstruction of 2-D multiband signals,” *Proc. 3rd IEEE Int. Conf. Image Process.*, vol. I, pp. 701–704, Sept. 1996.

[29] R. Venkataramani and Y. Bresler, “Perfect reconstruction formulae and bounds on aliasing error in sub-Nyquist nonuniform sampling of multiband signals,” *IEEE Trans. Inform. Theory*, vol. 46, pp. 2173–2183, Sept. 2000.

[30] ——, “Optimal sub-Nyquist nonuniform sampling and reconstruction of multiband signals,” *IEEE Trans. Signal Processing*, vol. 49, pp. 2301–2313, Oct. 2001.

[31] H. Landau, “Necessary density conditions for sampling and interpolation of certain entire functions,” *Acta Math.*, vol. 117, pp. 37–52, 1967.

[32] D. Seidner and M. Feder, “Vector sampling expansions,” *IEEE Trans. Signal Processing*, vol. 48, pp. 1401–1416, May 2000.

[33] P. Moulin, M. Anitescu, K. O. Kortanek, and F. A. Potra, “The role of linear semi-infinite programming in signal-adapted QMF bank design,” *IEEE Trans. Signal Processing*, vol. 45, pp. 2160–2174, Sept. 1997.

[34] R. G. Shenoy, D. Burnside, and T. W. Parks, “Linear periodic systems and multirate filter design,” *IEEE Trans. Signal Processing*, vol. 42, pp. 2242–2255, Sept. 1994.

[35] E. J. Anderson and P. Nash, *Linear Programming in Infinite-Dimensional Spaces*. Chichester, U.K.: Wiley, 1987.

[36] R. M. Young, *An Introduction to Nonharmonic Fourier Analysis*. New York: Academic, 2001.

[37] J. R. Higgins, *Sampling Theory in Fourier and Signals Analysis Foundations*. New York: Oxford, 1996.

[38] J. Benedetto and P. J. S. G. Ferreira, Eds., *Modern Sampling Theory: Mathematics and Applications*. Boston, MA: Birkhäuser, 2001.

[39] B. Vrcelj and P. P. Vaidyanathan, “MIMO biorthogonal partners and applications,” *IEEE Trans. Signal Processing*, vol. 50, pp. 528–543, Mar. 2002.

[40] X. G. Xia, “New precoding for intersymbol interference cancellation using nonmaximally decimated multirate filterbanks with ideal FIR equalizers,” *IEEE Trans. Signal Processing*, vol. 50, pp. 2431–2441, Oct. 1997.

[41] C. A. Berenstein and D. C. Struppa, “On explicit solutions to the bezout equation,” *Syst. Contr. Lett.*, vol. 4, no. 1, pp. 33–39, Feb. 1984.

[42] R. Rajagopal and L. Potter, “MIMO FIR equalizers and orders,” *Proc. IEEE ICASSP*, vol. 4, pp. 2077–2080, May 2001.

[43] W. Rudin, *Principles of Mathematical Analysis*. New York: McGraw-Hill, 1976.

Raman Venkataramani (S’97–M’01) received the B.Tech. degree from the Indian Institute of Technology, Madras, India, in 1995, the M.S. degree from Johns Hopkins University, Baltimore, MD, in 1997, and the Ph.D. degree from the University of Illinois at Urbana-Champaign in 2001, all in electrical engineering.

From November 2001 to December 2002, he was a member of technical staff at Bell Laboratories, Lucent Technologies, Murray Hill, NJ. From January 2003 to September 2003, he was a research associate at Harvard University, Cambridge, MA. He is now a post-doctoral researcher with Seagate Technology, Pittsburgh, PA. His interests include information theory, communications, statistical signal processing, signal sampling and interpolation, and inverse problems.

Yoram Bresler (F’99) received the B.Sc. (cum laude) and M.Sc. degrees from the Technion, Israel Institute of Technology, Haifa, Israel, in 1974 and 1981, respectively, and the Ph.D. degree from Stanford University, Stanford, CA, in 1986, all in electrical engineering.

From 1974 to 1979, he served as an electronics engineer in the Israeli Defense Force. From 1985 to 1987, he was a Research Associate with the Information Systems Laboratory, Stanford University, working on sensor array processing and medical imaging. In 1987, he joined the University of Illinois at Urbana-Champaign, where he is currently a Professor with the Department of Electrical and Computer Engineering and the Bioengineering Program and a Research Professor with the Coordinated Science Laboratory. From 1995 to 1996, he was a visiting Associate Professor at the Technion, Israel Institute of Technology, Haifa, Israel. His current research interests include multidimensional and statistical signal processing and their applications to inverse problems in imaging. He is on the editorial board of *Machine Vision and Applications*.

Dr. Bresler was an Associate Editor of the IEEE TRANSACTIONS ON IMAGE PROCESSING from 1992 to 1993 and a member of the IEEE Image and Multidimensional Signal Processing Technical Committee from 1994 to 1998. He is currently on the Awards Board of the IEEE Signal Processing Society (SPS). In 1988 and 1989, he received the Senior Paper Awards from the SPS, and a paper he coauthored with one of his students received the Young Author Award from the same society in 2002. He received a 1991 NSF Presidential Young Investigator Award, the Technion Fellowship in 1995, and the Xerox Senior Award for Faculty Research in 1998. He was named a University of Illinois Scholar in 1999 and appointed as an Associate at the Center for Advanced Study of the University for 2001 to 2002.