A FAMILY OF EQUALIZERS FOR OPTIMAL SEQUENCE DETECTION

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ABSTRACT

We present a family of equalizers and targets for certain inter-symbol interference (ISI) channels with the property that the performance of a maximum-likelihood (ML) or maximum *a posteriori* (MAP) based detector for the target channel is also simultaneously optimal for the equalized channel. In particular, the MMSE Decision feedback equalizer (DFE) belongs to this family of filters. Although, these solutions are infinite impulse response (IIR) filters, we can achieve good performance using finite impulse response (FIR) filters. We present an algorithm for designing equalizers and FIR targets that minimize the probability of sequence detection error.

1. INTRODUCTION

Communication systems that can be modeled as inter-symbol interference (ISI) channels often employ the Viterbi algorithm to perform maximum-likelihood (ML) or maximum *a posteriori* (MAP) detection of the input sequence. The implementation of the Viterbi algorithm with low complexity requires that the ISI length be short. In practice the ISI is usually reduced by using linear equalization. The equalizer is a filter that modifies the channel impulse response to match a specified short FIR filter called the *partial response target*. The Viterbi detector operates on the equalized channel with the assumption that the target filter approximates the channel response well.

The purpose of this work is to identify the set of equalizers and targets that yield the best performance with respect to ML and MAP-based detection for the equalized channel. We first show the existence of a family of IIR filters that incur no loss of optimality. We relate these solutions to minimum mean-square (MMSE) decision feedback equalizers (DFE). We finally extend these ideas to the FIR case and show how to design optimal FIR targets.

1.1. Definitions and Notation

We denote a discrete-time sequence $\{a_n : n \in \mathbb{Z}\}$ by $\boldsymbol{a}$. The discrete-time Fourier transform of a finite energy sequence $\boldsymbol{a}$ is defined as

$$\mathcal{F}\{\boldsymbol{a}\} = A(\omega) = \sum_n a_n e^{-j2\pi n\omega}.$$

The convolution of sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{c} = \boldsymbol{a} \star \boldsymbol{b}$ and their inner product is $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_n a_n^* b_n$, where $*$ denotes complex conjugation. Thus, the norm of $\boldsymbol{a}$ is $\|\boldsymbol{a}\| = \langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2}$. Given a sequence $\boldsymbol{a}$, let $\ddot{\boldsymbol{a}}$ be the sequence obtained by time-reversal and conjugation of $\boldsymbol{a}$, i.e., $\ddot{a}_n = a_{-n}^*$. For a real or circularly symmetric complex stationary random process, $\boldsymbol{x}$, denote its auto-correlation function by $r_n^x = \mathsf{E}(x_{m+n} x_m^*)$ where $\mathsf{E}(\cdot)$ denotes expectation. The power spectral density of this random process is $S_x(\omega) = \mathcal{F}\{\boldsymbol{r}^x\}$. Finally, let $\Re(\cdot)$ denote the real part.

1.2. Channel Model

We model our ISI channel as a real or complex discrete-time linear time invariant system

$$\boldsymbol{y} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{x} = \{x_m\}$ is the input to the channel, $\boldsymbol{h} = \{h_m\}$ is the channel response and $\boldsymbol{w} = \{w_n\}$ is an additive white Gaussian noise with power spectral density $S_w(\omega) = \sigma_w^2$. Assume that the channel impulse response has finite energy but is possibly non-causal and IIR. Equation (1) describes a variety of communication systems including magnetic recording.

In this paper, we assume that the input symbols are independent and identically distributed (IID) with $S_x(\omega) = 1$. In the case of complex channels, the noise is assumed to be a circularly symmetric, i.e., the real and imaginary components of the noise samples are independent with variance $\sigma_w^2/2$.

2. MAXIMUM-LIKELIHOOD SEQUENCE DETECTION

Suppose that a message $\boldsymbol{x} = \{x_m : m = 0, \ldots, M-1\}$ is transmitted through the channel (1). The received signal is given by

$$y_n = \sum_{m=0}^{M-1} h_{n-m} x_m + w_n. \tag{2}$$

Given the output sequence $\boldsymbol{y}$, the ML sequence detector estimates the input sequence as $\hat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} p(\boldsymbol{y}|\boldsymbol{x})$. Since the additive noise is white Gaussian, this reduces to finding the input whose channel response has minimum Euclidean distance from the output:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} D(\boldsymbol{y}, \boldsymbol{x}) \tag{3}$$

where

$$D(\boldsymbol{y}, \boldsymbol{x}) := \sum_n \Big| y_n - \sum_{m=0}^{M-1} h_{n-m} x_m \Big|^2. \tag{4}$$

The above summation is carried over the finite region of interest where the samples y_n are available. If the ISI is small, i.e., h_n is a short FIR sequence, the above problem cost function can be minimized exactly in a computationally efficient way using the Viterbi algorithm [1]. For long ISI lengths the usefulness of the above approach is limited because the complexity grows exponentially with the length of the ISI.

3. REVIEW OF LINEAR EQUALIZATION

Let $\boldsymbol{f} = \{f_n\}$ denote the equalizer filter and $\boldsymbol{g} = \{g_n\}$ denote the target filter. The equalizer output is

$$\boldsymbol{z} = \boldsymbol{f} \star \boldsymbol{y} = \boldsymbol{f} \star \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{f} \star \boldsymbol{w} = \boldsymbol{l} \star \boldsymbol{x} + \boldsymbol{u} \tag{5}$$

where $\boldsymbol{l} = \boldsymbol{f} \star \boldsymbol{h}$ is the response of the equalized channel and $\boldsymbol{u} = \boldsymbol{f} \star \boldsymbol{w}$ is the output noise whose power spectral density is $S_u(\omega) = |F(\omega)|^2 S_w(\omega) = \sigma_w^2 |F(\omega)|^2$. Traditionally, the equalized channel response $\boldsymbol{l}$ and the target $\boldsymbol{g}$ are designed to be "close" to each other, while keeping the overall noise as white as possible.

3.1. Zero Forcing Equalizer (ZFE)

A zero forcing equalizer modifies the channel response to match a given target filter exactly, i.e., $\boldsymbol{l} = \boldsymbol{g}$. Thus, the equalizer is given by $F(\omega) = G(\omega)/H(\omega)$. The spectral density of the noise $\boldsymbol{u}$ is

$$S_u(\omega) = |F(\omega)|^2 S_w(\omega) = \frac{|G(\omega)|^2}{|H(\omega)|^2} \sigma_w^2. \tag{6}$$

An undesirable problem with zero forcing is that if $|H(\omega)|$ has a spectral null or attains very small values, the equalized noise is highly colored and amplified. For this reason, the ZFE is rarely used in practice.

3.2. Minimum Mean Squared Error (MMSE) Equalizer

A widely used equalizer in practical systems is the MMSE equalizer, which is designed to minimize the variance of the equalization error $\boldsymbol{e} = \boldsymbol{f} \star \boldsymbol{y} - \boldsymbol{g} \star \boldsymbol{x}$. The expression for the MMSE equalizer is given by

$$F(\omega) = \frac{S_x(\omega) H^*(\omega) G(\omega)}{|H(\omega)|^2 S_x(\omega) + \sigma_w^2}. \tag{7}$$

The spectral density of the the estimation error is given by

$$S_e(\omega) = \frac{|G(\omega)|^2 S_x(\omega) \sigma_w^2}{|H(\omega)|^2 S_x(\omega) + \sigma_w^2} \tag{8}$$

The advantage of the MMSE design over the ZFE is that we do not have stability issues due to any spectral nulls in $H(\omega)$. The MMSE error (8) is less colored and smaller than the ZFE noise (6). However, the MMSE error is signal dependent and non-Gaussian in general. This may cause undesirable effects during Viterbi detection.

3.3. Target Design

Instead of choosing a fixed target, we seek the best target of a given length. In practice, the target design is usually done in conjunction with MMSE equalization. Since $S_x(\omega) = 1$, we minimize the variance of the MMSE equalization error (8):

$$\min \mathcal{E} = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_e(\omega) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G(\omega)|^2 \sigma_w^2 d\omega}{|H(\omega)|^2 + \sigma_w^2}$$

assuming a finite length target: $\boldsymbol{g} = \{g_0, g_1, \ldots, g_{L-1}\}$. The resulting cost function is a simple quadratic function of the target filter taps. To avoid the trivial solution $\boldsymbol{g} = 0$, we impose additional constraint on $\boldsymbol{g}$ such as the *unit-energy constraint*: $\|\boldsymbol{g}\|^2 = 1$ or the *monic constraint*: $g_0 = 1$. In each of these cases, the optimal target is found easily by minimizing the cost function subject to the appropriate constraint. The solutions to these problems for the design of FIR equalizers and targets can be found in [2].

For illustrative purposes, we present the solution to the monic design in the IIR case, where the problem can be expressed in the frequency domain as

$$\min \mathcal{E} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G(\omega)|^2 \sigma_w^2 d\omega}{|H(\omega)|^2 + \sigma_w^2} \tag{9}$$

over all causal targets $\boldsymbol{g}$ with $g_0 = 1$. Using variational calculus, it can be shown that the solution is

$$Q(\omega) := |G(\omega)|^2 = \lambda(|H(\omega)|^2 + \sigma_w^2) \tag{10}$$

where

$$\lambda = \exp\Big(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|H(\omega)|^2 + \sigma_w^2) d\omega \Big).$$

The target $G(\omega)$ can be chosen as the causal minimum-phase spectral factor of $Q(\omega)$. The MMSE equalizer (7) reduces to

$$F(\omega) = \frac{H^*(\omega) G(\omega)}{|H(\omega)|^2 + \sigma_w^2}. \tag{11}$$

Coincidentally, this is related to the linear MMSE decision feedback equalizer (DFE) for the given ISI channel [3, 4] with $\boldsymbol{g}$ being the feedback filter and $\boldsymbol{f}$ being the combination of the matched filter and the feed-forward filter. We emphasize that the sequence detector need not be implemented using decision feedback, but simply that the filters have the same form as the optimal MMSE DFE. The spectrum of the estimation error (8) is white for this solution. It has been experimentally observed the MMSE DFE design attains better performance than other design criteria. We shall formally prove this conjecture below.

4. SEQUENCE DETECTION FOR THE EQUALIZED CHANNEL

Let us consider the performance of ML sequence detection for the equalized channel (5):

$$\boldsymbol{z} = \boldsymbol{l} \star \boldsymbol{x} + \boldsymbol{f} \star \boldsymbol{w}.$$

Suppose that the above channel approximates the so-called *target channel* whose response is the target filter $\boldsymbol{g}$:

$$\tilde{\boldsymbol{z}} = \boldsymbol{g} \star \boldsymbol{x} + \tilde{\boldsymbol{u}} \tag{12}$$

where $\tilde{\boldsymbol{u}}$ white Gaussian noise. The ML sequence detector for this channel takes the form

$$\hat{x} = \arg\min_{\boldsymbol{x}} \tilde{D}(\tilde{\boldsymbol{z}}, \boldsymbol{x}) \tag{13}$$

where the surrogate cost function is

$$\tilde{D}(\tilde{\boldsymbol{z}}, \boldsymbol{x}) = \sum_n \left| \tilde{z}_n - \sum_m g_{n-m} x_m \right|^2. \tag{14}$$

Now, consider the following problem

$$\hat{x} = \arg\min_{\boldsymbol{x}} \tilde{D}(\boldsymbol{z}, \boldsymbol{x}) \tag{15}$$

where $\boldsymbol{z}$ is the output of the equalized channel (5), rather than the target channel (12). In other words, we are minimizing the cost function $\tilde{D}$ applied to the output of the equalized channel. We would like to study the performance loss due to this modification relative to the optimal rule (3). Our main result is summarized below.

Theorem 1. *Suppose the equalizer $\boldsymbol{f}$ and target $\boldsymbol{g}$ are chosen such that*

$$F(\omega) = \frac{H^*(\omega)G(\omega)}{|H(\omega)|^2 + \beta} \tag{16}$$

$$|G(\omega)|^2 = \alpha(|H(\omega)|^2 + \beta) \tag{17}$$

for any $\alpha > 0$ and $\beta > -\inf_\omega |H(\omega)|^2$, then

$$\tilde{D}(\boldsymbol{z}, \boldsymbol{x}) - \|\boldsymbol{z}\|^2 = \alpha(D(\boldsymbol{y}, \boldsymbol{x}) - \|\boldsymbol{y}\|^2) + \alpha\beta\|\boldsymbol{x}\|^2.$$

In particular,

$$\arg\min_{\boldsymbol{x}} D(\boldsymbol{y}, \boldsymbol{x}) = \arg\min_{\boldsymbol{x}} (\tilde{D}(\boldsymbol{z}, \boldsymbol{x}) - \alpha\beta\|\boldsymbol{x}\|^2) \tag{18}$$

since $\boldsymbol{y}$ and $\boldsymbol{z}$ are constants in the minimization.

The proof is very straightforward. A corollary is that if all the input sequences in the message codebook have equal energy, the solutions to the problems (3) and (15) are identical:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} D(\boldsymbol{y}, \boldsymbol{x}) = \arg\min_{\boldsymbol{x}} \tilde{D}(\boldsymbol{z}, \boldsymbol{x}). \tag{19}$$

For example, if the input symbols are elements of the Q-phase PSK constellation, i.e., $x_n \in \mathcal{X} = \{e^{j2\pi q/Q} : q = 0, \ldots, Q-1\}$ in the complex case or the BPSK constellation $\mathcal{X} = \{-1, +1\}$ in the real case, then all messages sequences have equal energy. Thus, (19) is immediately applicable. If the inputs have unequal energy, we need the energy bias term in (18).

Theorem 1 shows that for a special family of equalizer and target filters there is no performance loss if we minimize the surrogate cost function $\tilde{D}(\boldsymbol{z}, \boldsymbol{x})$ instead of the original cost $D(\boldsymbol{y}, \boldsymbol{x})$. In fact, the equalized channel (5) and the target channel (12) have the same *a posteriori* probability distribution. Thus, the two channels are equivalent as far as *any* MAP based detector is concerned. The parameter α is merely a scaling factor but β affects the shape of the filters. The phase response of $G(\omega)$ is arbitrary, but the most logical choice would be to choose $G(\omega)$ as the causal minimum-phase spectral factor of (17). In general, these solutions are IIR. However for a Viterbi-based implementation, we require a short FIR target. We address this problem in Section 5.

4.1. Special Cases

Letting $\alpha = 1$ and $\beta = 0$ we obtain $|G(\omega)|^2 = |H(\omega)|^2$ and $F(\omega) = G(\omega)/H(\omega)$, which corresponds to an all-pass zero-forcing equalizer filter for which the noise remains white. Setting $\alpha = \lambda$ and $\beta = \sigma_w^2$ yields the MMSE DFE solution (10) and (11). This proves that the monic design is optimal in the asymptotic (IIR) case. When $\beta \neq \sigma_w^2$, the solution corresponds to an MMSE DFE design for a different noise variance β. However, this mismatch causes no performance loss in sequence detection. Curiously, some negative values $-\inf_\omega |H(\omega)|^2 < \beta < 0$ are also allowed even though they do not represent the variance of any noise.

5. OPTIMAL FIR TARGET DESIGN

In the last section we derived a family of IIR equalizers and targets that achieve the optimal performance for sequence detection of the equalized channel. In practice, we can afford to use long equalizers, but we still require a short FIR target to keep the complexity of the Viterbi algorithm low. In this section, we present a solution to the problem of FIR target design for real channels with a BPSK input ($\mathcal{X} = \{-1, +1\}$) assuming that the equalizer is IIR.

Suppose that $\boldsymbol{x}^\circ$ is the actual input to the channel, and $\hat{\boldsymbol{x}}$ is the ML estimated sequence. Then $\boldsymbol{e} = (\hat{\boldsymbol{x}} - \boldsymbol{x}^\circ)$ is an *error event*. Of all such events, the *dominant error event* is that which minimizes $\|\tilde{e}\|^2$ where $\tilde{e} = \boldsymbol{h} \star \boldsymbol{e}$ is the dominant *output error event*. The following result is easily proved using error analysis similar to that of standard Viterbi detection [1].

Theorem 2. *The probability of sequence detection error at moderately high SNR for a real BPSK channel is given by* $P_e \simeq \kappa Q_g(\sqrt{\mathsf{SNR}})$ *for some constant* κ*, where* $Q_g(\cdot)$ *is the Gaussian* Q*-function and* $\mathsf{SNR} = \max_{\boldsymbol{v}} \mathsf{SNR}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{v})$ *is the effective signal-to-noise ratio of the system:*

$$\mathsf{SNR}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{v}) = \frac{|\Re\langle \boldsymbol{e}, \boldsymbol{p} \star \boldsymbol{h} \star \boldsymbol{e}\rangle|^2}{\|(\boldsymbol{q} - \ddot{\boldsymbol{p}} \star \ddot{\boldsymbol{h}}) \star \boldsymbol{e} - \boldsymbol{v}\|^2 + \sigma_w^2 \|\boldsymbol{p} \star \boldsymbol{e}\|^2},$$

where $\boldsymbol{p} = \boldsymbol{f} \star \ddot{\boldsymbol{g}}$*,* $\boldsymbol{q} = \boldsymbol{g} \star \ddot{\boldsymbol{g}}$*, and* $\boldsymbol{v}$ *is a sequence with the same support temporal as the dominant input error event* $\boldsymbol{e}$*.*

The result is valid for FIR as well as IIR filters. The bit error rate (BER) also takes the same form as P_e but has a different constant κ. For an optimal design, we seek the equalizer and target filters that maximize the value of SNR. The optimal equalizer $\boldsymbol{f}$ and target $\boldsymbol{g}$ are those that maximize SNR subject to relevant constraints. In particular we are interested in an FIR target of length L and an IIR equalizer. In this case, it is more convenient to perform the maximization over $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{v}$ since $\boldsymbol{f}$ and $\boldsymbol{g}$ can be recovered from $\boldsymbol{p}$ and $\boldsymbol{q}$ by spectral factorization. Note that $\boldsymbol{q}$ is the autocorrelation of the FIR target $\boldsymbol{g}$. Hence, $\boldsymbol{q}$ is FIR and constrained by $Q(\omega) \geq 0$.

Let $\boldsymbol{\delta}$ denote the discrete delta function: $\delta_n = 0$ for $n \neq 0$ and $\delta_0 = 1$. Then, the triplets $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{v})$ and $(\boldsymbol{p}', \boldsymbol{q}', \boldsymbol{v}') := (\boldsymbol{p}, \boldsymbol{q} + \beta\boldsymbol{\delta}, \boldsymbol{v} - \beta\boldsymbol{e})$ for any $\beta > 0$ produce the same value for the effective SNR. Thus, we can perform the maximization ignoring the constraint $Q(\omega) \geq 0$ and finally add a sufficiently large β to $Q(\omega)$ to make it nonnegative. Having rid of the constraint on $Q(\omega)$, the maximization is readily transformed into a quadratic minimization and solved analytically. We omit the details here. The above argument also shows that there are infinitely many solutions parameterized by β. This is reminiscent of the family of solutions in Theorem 1. The two families of solutions converge as $L \to \infty$.

5.1. Example

We illustrate the FIR target design using the example of a real ISI channel (1) with binary inputs $x_n \in \{-1, +1\}$ and impulse response $h_n = e^{-n/2}$ for $0 \leq n \leq 10$ and $h_n = 0$ otherwise. The SNR is defined as $\|\boldsymbol{h}\|^2/\sigma_w^2$ where σ_w^2 is the noise variance. For each SNR we design the optimal length-3 target and IIR equalizer which is truncated to 21-taps (centered at the origin) since it captures most of the energy. The dominant error event for this channel is $\boldsymbol{e} = \{1, -1\}$. We also design length-21 MMSE equalizers (centered at 0) and length-3 targets described in Section 3 (see also [2]) for the *unit-energy* and *monic* target constraints.

Using computer simulations we compare the three designs in terms of their BER performance for IID binary inputs. The three systems use the Viterbi algorithm to perform the sequence detection. The results are shown in Figure 1. It is clear that the optimal FIR design slightly outperforms the monic FIR design while the unit-energy design performs worst of all. The performance gap between the optimal and monic designs will diminish even further if we let the target length grow, as predicted by Theorem 1.

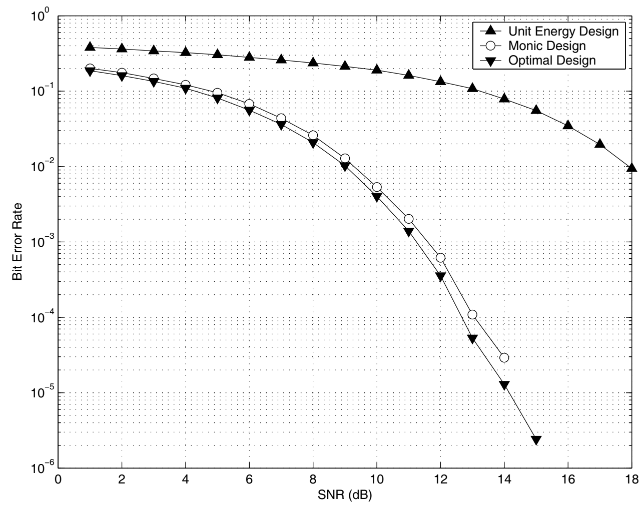

Fig. 1. Comparison of BER peformance of three designs.

6. CONCLUSION

We proved the existence of a family of IIR equalizers and targets for ML optimal sequence detection in certain ISI channels. The MMSE DFE is shown to be a particular solution in this family, proving the previous conjecture regarding its optimality. We also showed how to design optimal FIR targets for optimal detection.

7. REFERENCES

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