Bounds on the Achievable Region for Certain Multiple Description Coding Problems†

Raman Venkataramani
Univ. of Illinois at
Urbana-Champaign
Urbana, IL 61801, U.S.A.
raman@ifp.uiuc.edu

Gerhard Kramer
Bell Labs, Lucent Technologies
600 Mountain Avenue
Murray Hill, NJ 07974, U.S.A.
gkr@bell-labs.com

Vivek K Goyal
Digital Fountain
39141 Civic Center Drive
Fremont, CA 94538, U.S.A.
v.goyal@ieee.org

Abstract — **An achievable region for the L-channel multiple description coding problem is presented. This region generalizes previous two-channel results of El Gamal and Cover and of Zhang and Berger. New outer bounds on the rate distortion region for memoryless Gaussian sources with mean-squared error distortion are also derived. For the Gaussian source, the achievable region meets the outer bound at certain points.**

I. Problem Description

Consider a source that emits a sequence $X^N = X^{(1)}, X^{(2)}, \dots, X^{(N)}$ of N independent and identically distributed (i.i.d.) random variables. X^N is encoded into L *descriptions* $J_1, J_2, \dots, J_L$ at rates $R_1, R_2, \dots, R_L$ nats per source symbol. Suppose that each description is either transmitted error-free or lost completely. Thus the receiver encounters one of 2^L configurations depending on which descriptions are received. Excepting the trivial case where no description is received, we can represent the receiver as a collection of $2^L - 1$ decoders, where each decoder produces an output based on a non-empty subset of $\{J_1, \dots, J_L\}$.

Let $\mathcal{L} = \{1, \dots, L\}$ and let $2^{\mathcal{L}}$ be its power set. For every $\mathcal{K} \in 2^{\mathcal{L}}$, let $X_{\mathcal{K}}^N = X_{\mathcal{K}}^{(1)}, \dots, X_{\mathcal{K}}^{(N)}$ denote the output of the decoder whose inputs are $\{J_k : k \in \mathcal{K}\}$. Next let $d_{\mathcal{K}} = E\big[\frac{1}{N}\sum_{n=1}^N \delta_{\mathcal{K}}\big(X^{(n)}, X_{\mathcal{K}}^{(n)}\big)\big]$ denote the expected distortion per source symbol associated with the output $X_{\mathcal{K}}^N$, where $\delta_{\mathcal{K}}(\cdot,\cdot)$ is a distortion measure. Our problem is to find the set of rates $\{R_1, \dots, R_L\}$ and distortions $\{d_{\mathcal{K}} : \mathcal{K} \in 2^{\mathcal{L}} - \{\emptyset\}\}$ that are achievable in the usual Shannon sense. We call this region the *rate-distortion (RD) region*.

II. An Achievable Region

The set difference between collections of sets $\mathcal{C}$ and $\mathcal{D}$ is denoted $\mathcal{C} - \mathcal{D} = \{\mathcal{M} \in \mathcal{C} : \mathcal{M} \notin \mathcal{D}\}$. Also, we write $R_{\mathcal{K}}$ as a shorthand for $\sum_{k\in\mathcal{K}} R_k$ and $X_{(\mathcal{C})}$ for a collection of random variables $\{X_{\mathcal{N}} : \mathcal{N} \in \mathcal{C}\}$. Our first result is an achievable region for the general L-description problem.

Theorem 1 *Let $X_{(2^{\mathcal{L}})}$ be 2^L finite-alphabet random variables jointly distributed with X. Then the RD region contains the rates and distortions satisfying*

$$\begin{aligned} d_{\mathcal{K}} &\geq E\delta_{\mathcal{K}}(X, X_{\mathcal{K}}) \\ R_{\mathcal{K}} &\geq (|\mathcal{K}|-1)I(X; X_{\emptyset}) - H(X_{(2^{\mathcal{K}})}|X) \\ &\quad + \sum_{\mathcal{M}\subseteq\mathcal{K}} H(X_{\mathcal{M}}|X_{(2^{\mathcal{M}}-\{\mathcal{M}\})}) \end{aligned}$$

for every $\mathcal{K} \in 2^{\mathcal{L}} - \{\emptyset\}$, where $|\mathcal{K}|$ is the cardinality of $\mathcal{K}$.

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In Theorem 1, $X_\emptyset$ is an arbitrary random variable. For $L = 2$, this result generalizes the result of Zhang and Berger [1]. Additionally, with $X_\emptyset$ set to a constant, e.g. 0, it reduces to the result of El Gamal and Cover [2].

Theorem 1 holds more generally for well-behaved continuous sources if all entropies $H(\cdot)$ are replaced by differential entropies $h(\cdot)$. We next focus exclusively on the Gaussian source with mean-squared error distortion.

III. The Quadratic Gaussian Case

The following outer bound generalizes a result of Ozarow [3]. For this result, a collection of M disjoint sets $\{\mathcal{K}_m\}_{m=1}^M$ is called a *partition* of a set $\mathcal{K}$ if $\bigcup_{m=1}^M \mathcal{K}_m = \mathcal{K}$.

Theorem 2 *For each $\mathcal{K} \in 2^{\mathcal{L}}$, the achievable rates R_ℓ, $\ell \in \mathcal{L}$, and distortions $d_{\mathcal{K}}$, $\mathcal{K} \in 2^{\mathcal{L}}$, satisfy*

$$e^{-2R_{\mathcal{K}}} \leq \min_{\{\mathcal{K}_m\}_{m=1}^M} \inf_{\lambda\geq 0} \left(d_{\mathcal{K}} \frac{\prod_{m=1}^M (d_{\mathcal{K}_m} + \lambda)}{(d_{\mathcal{K}} + \lambda)(1+\lambda)^{M-1}} \right)$$

where the minimization is performed over all partitions of $\mathcal{K}$.

Example Let the source emit unit-variance i.i.d. Gaussian random variables, and suppose that we care only about the reconstructions from individual descriptions $X_{\{1\}}, X_{\{2\}}, \dots, X_{\{L\}}$, and the reconstruction from all the descriptions $X_{\mathcal{L}}$. An important operating point is that of equal side distortions and rates on all channels: $d_{\{\ell\}} = d$ and $R_\ell = R \geq -\frac{1}{2}\log d$, $\ell \in \mathcal{L}$. For this operating point, we can show using Theorems 1 and 2, that the best central distortion achievable is

$$d_{\mathcal{L}} = \sup_{\lambda\geq 0} \left(\frac{e^{-2LR}\lambda(1+\lambda)^{L-1}}{(d+\lambda)^L - e^{-2LR}(1+\lambda)^{L-1}} \right).$$

In fact, we can show that the outer and inner bounds meet everywhere on an open set in the vicinity of this point.

References

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