Optimal Sampling and Reconstruction in Multiple-Input-Multiple-Output (MIMO) Systems[1]

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Abstract— **We consider a sampling scheme where a set of multiband input signals are passed through a MIMO liner time-invariant system and the outputs are sampled. MIMO sampling is a very general scheme that encompasses various other schemes, including Papoulis' generalized sampling and nonuniform sampling as special cases. We present necessary density conditions for stable MIMO sampling. These results generalize Landau's classical density results for stable sampling and interpolation. Under the additional assumption of periodic (but possibly nonuniform) sampling sets, we present necessary and sufficient conditions on the system and sampling rates for stable MIMO sampling.**

I. Introduction

Given a multiple-input multiple-output (MIMO) channel with observable outputs, the problem of multichannel deconvolution or multichannel separation of a convolutive mixture is to invert or equalize the channel to recover the channel inputs. Applications include multiuser or multiaccess wireless communications and space-time coding with antenna arrays, or telephone digital subscriber loops, multisensor biomedical signals, multi-track magnetic recording, multiple speaker (or other acoustic source) separation with microphone arrays, geophysical data processing, and multichannel image restoration.

To enable digital processing for the inversion of the channel, its continuous-time outputs are sampled prior to processing, and the goal is to reconstruct the continuous-time channel inputs. We assume that the channel characteristics are known (e.g., can be estimated accurately using known test input signals.) We call this problem *MIMO sampling*.

Our problem is formulated as follows. Let $x_r(t)$, $r = 1, \dots, R$, be a collection of complex-valued signals whose spectral supports are sets $\mathcal{F}_r \subseteq \mathbb{R}$ of finite measure. We refer to such signals as *multiband* signals because in practice $\mathcal{F}$ is a finite union of intervals. These R signals are input to a MIMO channel consisting of linear time-invariant filters to produce P outputs

$$y_p(t) = \sum_{r=1}^{R} g_{pr}(t) * x_r(t), \quad p = 1, \dots, P \qquad (1)$$

where $*$ denotes convolution, and $\{g_{pr}\}$ are square-integrable impulse responses. Each output $y_p(t)$ is subsequently sampled on a discrete set $\Lambda_p = \{\lambda_{np} : n \in \mathbb{Z}\}$ and these samples are then used to reconstruct the inputs. This sampling scheme is very general and subsumes various other sampling schemes as special cases. For instance Papoulis' generalized sampling [1] is essentially a single-input multiple-output (SIMO) sampling scheme, i.e., MIMO sampling with $R = 1$. An extension of Papoulis' sampling expansion to vector valued inputs [2] is also a special case with all inputs having identical low-pass spectra, i.e., $\mathcal{F}_r = [-B, B]$.

In this paper, we present an overview of our recent results on MIMO sampling [3–6]. In particular, we present necessary conditions [4] on $\{\Lambda_p\}$ and the channel for *stable reconstruction* of the inputs $x_r(t)$ from the MIMO output samples $\{y_p(\lambda_{np})\}$. Similar results are also available [4](but not reviewed here) for the dual problem of *consistent reconstruction*: necessary conditions on $\{\Lambda_p\}$, to ensure that $\exists x_r(t)$ such that $y_p(\lambda_{np}) = c_{np}$ for any sequence $\{c_{np} : n \in \mathbb{Z},\ p = 1, \dots, P\} \in l^2$. Under the additional assumption of periodic (but possibly non-uniform) sampling sets, we present conditions that are both necessary and sufficient [5]. Finally, we review results on optimum reconstruction under these conditions [6].

Landau [7] proved the following fundamental result for sampling of multiband signals. Let $x \in L^2(\mathbb{R}^d)$ be a continuous function whose Fourier transform is supported on a measurable set $\mathcal{F} \subset \mathbb{R}^d$. Then, for stable reconstruction of any such x from its samples, it is necessary that the density of Λ be no less than the measure of $\mathcal{F}$. We refer to this problem as classical sampling.

Gröchenig and Razafinjatovo [8] provided a simpler proof of Landau's classical result for the case that $\mathcal{F}$ has zero boundary measure. We extended the idea of [8] (removing the restriction of zero boundary measure) to derive necessary density results for MIMO sampling of multiband signals [4]. Our results for single variate functions ($d = 1$) easily extend to multivariate functions.

For stable sampling, we prove that a family of $2^P - 1$ bounds hold—a lower bound on the joint lower density of each nonempty set of P output sampling sets. These bounds generalize Landau's necessary density results for classical sampling. Since the MIMO sampling scheme is extremely general, and encompasses various sampling schemes such as Papoulis' generalized sampling, and multicoset or periodic nonuniform sampling as special cases, we automatically have necessary conditions for all these sampling schemes as well.

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II. Preliminaries

Let

$$\mathcal{B}(\mathcal{F}) = \{x \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : X(f) = 0,\ \forall f \notin \mathcal{F}\} \quad (2)$$

denote the space of continuous $L^2(\mathbb{R})$ signals bandlimited to a measurable $\mathcal{F} \subseteq \mathbb{R}$, where $X(f)$ is the Fourier transform of signal $x(t)$. Let $\chi(\cdot)$ denote the indicator function, Let $\emptyset$ denote the empty set, $\mathcal{S}^c$, the complement of a set $\mathcal{S}$, and $|\mathcal{S}|$ its cardinality. Let $\mu(\mathcal{S})$ denote the Lebesgue measure and $\operatorname{int}\mathcal{S}$ and $\overline{\mathcal{S}}$ the interior and closure of a set $\mathcal{S} \subseteq \mathbb{R}$, respectively.

For any complex matrix $\boldsymbol{A}$, let $\boldsymbol{A}^H$ denote its conjugate-transpose, and $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$, the submatrix of $\boldsymbol{A}$ whose rows and columns are indexed by sets $\mathcal{R}$ and $\mathcal{C}$ respectively. If all rows (or columns) are chosen, we use "$\bullet$" in place of $\mathcal{R}$ (or $\mathcal{C}$).

A. Sampling Density

Let $\text{ext}^{\pm}$ denote "sup" and "inf". Then the following definitions generalize the notions of upper and lower densities [7, 9] to collections of sampling sets.

Definition 1: The *joint upper and lower densities* of a collection of discrete sets Λ_p, $p \in \mathcal{P}$ are defined as

$$D^{\pm}(\Lambda_1, \ldots, \Lambda_P) = \lim_{\gamma \to \infty} \text{ext}^{\pm}\, \nu_\gamma^{\pm}(\Lambda_1, \ldots, \Lambda_P)/(2\gamma), \quad (3)$$

where $\nu_\gamma^{\pm}(\Lambda_1, \ldots, \Lambda_P) = \text{ext}^{\pm}{}_{\tau} \sum_{p=1}^{P} |\Lambda_p \cap [\tau-\gamma, \tau+\gamma]|$ are the maximum and minimum number of points of the collection $\{\Lambda_p : p = 1, \ldots, P\}$ found in any interval of length 2γ.

If the lower and upper densities of $\{\Lambda_1, \ldots, \Lambda_P\}$ coincide, then the collection has *uniform* joint density of $D(\Lambda_1, \ldots, \Lambda_P) = D^{\pm}(\Lambda_1, \ldots, \Lambda_P)$. If each Λ_p has uniform density, then so does the collection $\{\Lambda_1, \ldots, \Lambda_P\}$, i.e., $D(\Lambda_1, \ldots, \Lambda_P) = \sum_{p=1}^{P} D(\Lambda_p)$.

B. Stable sampling

Expressing (1) in vector form, $\boldsymbol{y}(t) = \boldsymbol{g}(t) * \boldsymbol{x}(t)$, the components of the input vector $\boldsymbol{x}$ are multiband signals $x_r \in \mathcal{B}(\mathcal{F}_r)$, $\boldsymbol{y}$ is the channel output, and $\boldsymbol{g}$ is a matrix with entries $g_{pr}(t)$. Denote the corresponding operator by G. The space of inputs $\boldsymbol{x} \in \mathcal{H} \stackrel{\text{def}}{=} \mathcal{B}(\mathcal{F}_1) \times \cdots \times \mathcal{B}(\mathcal{F}_R)$ is a separable Hilbert space with its inner product defined as $\langle \boldsymbol{x}, \boldsymbol{w} \rangle = \int_{\mathbb{R}} \boldsymbol{w}^H(t)\boldsymbol{x}(t)dt$, $\boldsymbol{x}, \boldsymbol{w} \in \mathcal{H}$.

Let $\mathcal{R} = \{1, \ldots, R\}$ and $\mathcal{P} = \{1, \ldots, P\}$ denote index sets for the input and output components.

Definition 2: A collection of discrete sampling sets $\Lambda_p = \{\lambda_{np} : n \in \mathbb{Z}\}$, $p \in \mathcal{P}$ is said to be *stable* with respect to $(G, \mathcal{H})$ if there exist $A, B > 0$ such that

$$A\|\boldsymbol{x}\|^2 \le \sum_{p=1}^{P} \sum_{n \in \mathbb{Z}} |y_p(\lambda_{np})|^2 \le B\|\boldsymbol{x}\|^2 \quad (4)$$

for every $\boldsymbol{x} \in \mathcal{H}$, where $\boldsymbol{y} = G\boldsymbol{x}$. We sometimes refer to this as a *collection of stable MIMO sampling.*

This definition generalizes that for simple multiband sampling [9], and corresponds to the requirement that the linear operator mapping signals in $\mathcal{H}$ to the samples of the channel output be a bounded linear operator, and have a bounded inverse.

III. Necessary Density Conditions

Define the index set of inputs "active" at frequency f,

$$\mathcal{C}_f = \{r : f \in \mathcal{F}_r\}. \quad (5)$$

Then it is clear that $\boldsymbol{X}_{\mathcal{C}_f}(f)$ contains all the nonzero elements of $\boldsymbol{X}(f)$. Hence, the channel output can be expressed in the frequency domain as

$$\boldsymbol{Y}(f) = \boldsymbol{G}(f)\boldsymbol{X}(f) = \boldsymbol{G}_{\bullet,\mathcal{C}_f}(f)\boldsymbol{X}_{\mathcal{C}_f}(f) \quad (6)$$

where $\boldsymbol{G}(f)$ (the Fourier transform of $\boldsymbol{g}(t)$) is called the *channel frequency response matrix.*

Theorem 1: Suppose that $\mathcal{F}_r$, $r \in \mathcal{R}$ are real sets of finite measure, $\mathcal{H} = \mathcal{B}(\mathcal{F}_1) \times \cdots \times \mathcal{B}(\mathcal{F}_R)$, and Λ_p, $p \in \mathcal{P}$ are discrete sets with $D^+(\Lambda_p) < \infty$ that constitute a collection of stable sampling with respect to $(G, \mathcal{H})$. Then, for every $\Pi \subseteq \mathcal{P}$,

$$D^-(\{\Lambda_p : p \in \Pi\}) \ge \sum_{r=1}^{R} \mu(\mathcal{F}_r) - \int_{\mathbb{R}} \operatorname{rank}\left(\boldsymbol{G}_{\Pi^c,\mathcal{C}_f}(f)\right) df \quad (7)$$

where $\mathcal{C}_f = \{r : f \in \mathcal{F}_r\}$ and Π^c is the complement of Π in $\mathcal{P}$. Furthermore, if

$$\operatorname*{ess\,inf}_{f \in \mathcal{F}} \sigma_{\min}\left(\boldsymbol{G}_{\Pi^c,\mathcal{C}_f}(f)\right) = 0, \quad \mathcal{F} = \bigcup_{r \in \mathcal{R}} \mathcal{F}_r \quad (8)$$

for some $\Pi \neq \mathcal{P}$, then (7) is a strict inequality.

Theorem 1 provides a total of $2^P - 1$ lower bounds – one on the joint densities of each sub-collection of $\{\Lambda_p\}$. In particular, letting $\Pi = \mathcal{P}$ in (7), we obtain

$$D^-(\Lambda_1, \ldots, \Lambda_P) \ge \sum_{r=1}^{R} \mu(\mathcal{F}_r). \quad (9)$$

In other words, the combined sampling density on all the outputs must be no less than the combined bandwidth of all the input signals, which represents the total number of degrees of freedom per unit time contained in the inputs.

We can interpret these bounds as follows. Suppose that the outputs $y_p(t)$, $p \in \Pi^c$ are completely known for all $t \in \mathbb{R}$, which is the case that demands the weakest conditions from the $\{\Lambda_p : p \in \Pi\}$ for stable sampling. Then, $\boldsymbol{Y}_{\Pi^c}(f) = \boldsymbol{G}_{\Pi^c,\mathcal{C}_f}(f)\boldsymbol{X}_{\mathcal{C}_f}(f)$ is known for all f. Therefore, $\operatorname{rank}\left(\boldsymbol{G}_{\Pi^c,\mathcal{C}_f}(f)\right)$ is the number of independent components of $\boldsymbol{X}(f)$ at frequency f that can be determined from knowledge of $\boldsymbol{Y}_{\Pi^c}(f)$ alone. Consequently,

$$\int_{\mathbb{R}} \operatorname{rank}\left(\boldsymbol{G}_{\Pi^c,\mathcal{C}_f}(f)\right)$$

is the number of degrees of freedom per unit time in the inputs that can be resolved by knowing the outputs $y_p(t)$, $p \in \Pi^c$ completely (for all t). Therefore, the difference in right-hand side of (7) is the number of unresolved degrees of freedom per unit time in the inputs. The left-hand side of (7) is the joint lower density of $\{\Lambda_p :\ p \in \Pi\}$, i.e., the smallest local sampling density (number of samples per unit time in a local sense) contained in these sampling sets. Thus, (7) merely states that we require more samples than the unresolved degrees of freedom in the inputs (locally per unit time) for each choice of Π.

Note that this bound depends only on the submatrix of $\boldsymbol{G}(f)$ whose rows are indexed by the complement of Π and columns by $\mathcal{C}_f$ because $X_r(f)$ vanishes outside $\mathcal{F}_r$.

Next, if some singular value of $\boldsymbol{G}_{\Pi^c,\mathcal{C}_f}(f)$ takes arbitrarily small nonzero values, then we cannot stably invert $\boldsymbol{Y}_{\Pi^c}(f) = \boldsymbol{G}_{\Pi^c,\mathcal{C}_f}(f)\boldsymbol{X}_{\mathcal{C}_f}(f)$ to stably recover the independent components of $\boldsymbol{X}(f)$ and the density of Λ_p must be strictly larger than the right-hand side of (7).

Theorem 1 leads to the following simple necessary conditions on the *admissibility* of subsets of the continuous-time channel outputs for stable recovery of the inputs. Let $\mathcal{F} = \bigcup_{r\in\mathcal{R}} \mathcal{F}_r$.

Definition 3: A set of outputs y_p, $p \in \Pi$, $\Pi \subseteq \mathcal{P}$ is said to be an *admissible* set of outputs for $\mathcal{H}$ if

$$\operatorname*{ess\,inf}_{f\in\mathcal{F}} \lambda_{\min}\big(\boldsymbol{G}^H_{\Pi,\mathcal{C}_f}(f)\boldsymbol{G}_{\Pi,\mathcal{C}_f}(f)\big) > 0. \tag{10}$$

It is easily verified that (10), which states that the singular values of $\boldsymbol{G}_{\Pi,\mathcal{C}_f}(f)$ are uniformly bounded away from zero, is a necessary condition for stable recovery of $\boldsymbol{x} \in \mathcal{H}$ from the continuous-time outputs $\{y_p(t) : t \in \mathbb{R},\ p \in \Pi\}$.

Corollary 1: Under the hypotheses of Theorem 1 Π^c is an admissible output set for $\mathcal{H}$ for every $\Pi \subset \mathcal{P}$, $\Pi \neq \mathcal{P}$ for which $D^-(\{\Lambda_p : p \in \Pi\}) = 0$. In particular,

$$\operatorname*{ess\,inf}_{f\in\mathcal{F}} \lambda_{\min}\big(\boldsymbol{G}^H_{\bullet,\mathcal{C}_f}(f)\boldsymbol{G}_{\bullet,\mathcal{C}_f}(f)\big) > 0. \tag{11}$$

Equation (11), which states that the entire set $\mathcal{P}$ of outputs must be admissible for stable MIMO sampling is not surprising: even if all $y_p(t)$ are known for $t \in \mathbb{R}$, we cannot stably recover the channel inputs unless (11) holds. In fact, (11) yields an even simpler necessary condition:

$$P \geq |\mathcal{C}_f| \text{ a.e.}$$

i.e., the number of outputs must be no less than the number of overlapping input spectra at any frequency.

Next, suppose that $D^-(\{\Lambda_p : p \in \Pi\}) = 0$. Then, the output samples on the sampling sets $\{\Lambda_p : p \in \Pi\}$ are too sparse to contain any signal information. Therefore, we must rely entirely on the outputs samples taken on $\{\Lambda_p : p \in \Pi^c\}$ to achieve stable reconstruction, and an argument as before provides intuitive justification for the admissibility of Π^c.

The following result provides another necessary condition for stable sampling.

Theorem 2: Under the hypotheses of Theorem 1,

$$\operatorname*{ess\,sup}_{f\in\mathcal{F}} \sigma_{\max}\big(\boldsymbol{G}_{\Pi^+,\mathcal{C}_f}(f)\big) < \infty, \tag{12}$$

where $\Pi^+ = \{p \in \mathcal{P} : D^+(\Lambda_p) > 0\}$ and $\mathcal{F} = \bigcup_{r\in\mathcal{R}} \mathcal{F}_r$.

To interpret this result, note that whenever $D^+(\Lambda_p) = 0$ for some $p \in \mathcal{P}$, the samples of y_p on Λ_p are too sparse to provide any useful information. Thus, Π^+ can be viewed as the set of outputs whose samples are dense enough to provide information about the inputs. It follows that (12) is an implication of the upper stability bound in (4).

Clearly, $D^-(\Lambda_p) = D^+(\Lambda_p) = 0$ for all $p \notin \Pi^+$. Thus, $D^-(\{\Lambda_p : p \in (\Pi^+)^c\}) = 0$ and by Corollary 1, Π^+ must be an admissible set and (10) holds for $\Pi = \Pi^+$. In this case, Condition (10) applied to Π^+, and Condition (12) are necessary bounds on the smallest and largest singular values of the same matrix. So, if (12) does not hold then stable reconstruction from the sampled versions outputs will be impossible even if Π^+ is admissible. An example of this in the single-input single-output case is an integrator $\boldsymbol{G}(f) = 1/(j2\pi f)$.

Example 1: Consider a MIMO channel with $R = 2$ inputs, $P = 2$ outputs, and frequency response matrix $\boldsymbol{G}(f) = \left[\begin{smallmatrix} 1 & K(f) \\ 0 & 1 \end{smallmatrix}\right]$ where $K(f)$ is shown in Figure 1. Let $\mathcal{F}_1 = [-1, 1)$ and $\mathcal{F}_2 = [0, 2)$ be the input spectral supports. The input and output spectra for a typical set of channel inputs are illustrated in Figure 2. We interpret $Y_1(f)$ as the sum of the two pieces shown in the figure.

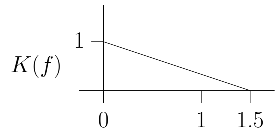

Fig. 1. $K(f)$

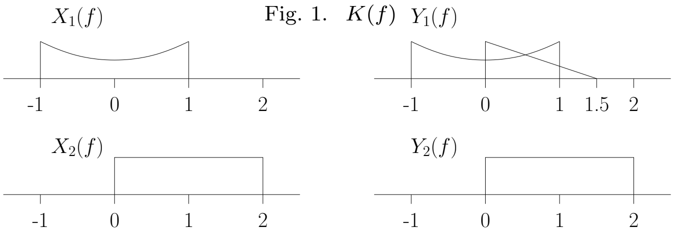

Fig. 2. Typical spectra of the channel inputs and outputs.

Let $y_1(t)$ and $y_2(t)$ be sampled on sets Λ_1 and Λ_2 respectively. Then, what are necessary conditions on Λ_1 and Λ_2 for stable MIMO sampling with respect to G?

We have $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 = [-1, 2]$ and

$$\boldsymbol{C}_f = \{r : f \in \mathcal{F}_r\} = \begin{cases} \{1\} & \text{if } f \in [-1, 0) \\ \{1, 2\} & \text{if } f \in [0, 1) \\ \{2\} & \text{if } f \in [1, 2) \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to check that (12) is satisfied regardless of Π^+. Also

$$\sigma_{\min}\big(\boldsymbol{G}_{\bullet,\mathcal{C}_f}(f)\big) = \frac{[2 + K^2(f)] - \sqrt{[2 + K^2(f)]^2 - 4}}{2}$$

is positive Hence, (11) is satisfied. Applying Theorem 1, we obtain the following density conditions:

$$\begin{aligned} &D^-(\Lambda_1, \Lambda_2) \geq \mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) = 4 \\ &\quad D^-(\Lambda_1) \geq \mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) - \int_{\mathbb{R}} \operatorname{rank}\big(\boldsymbol{G}_{2,\mathcal{C}_f}(f)\big) df \\ &\qquad\qquad = 4 - \int_{[-1,0]} 0 df - \int_{[0,1]} 1 df - \int_{[1,2]} 1 df = 2 \\ &\quad D^-(\Lambda_2) \geq \mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) - \int_{\mathbb{R}} \operatorname{rank}\big(\boldsymbol{G}_{1,\mathcal{C}_f}(f)\big) df \\ &\qquad\qquad = 1.5. \end{aligned}$$

Now, a simple calculation reveals that

$$\sigma_{\min}\big(\boldsymbol{G}_{1,\mathcal{C}_f}(f)\big) = \begin{cases} 1 & \text{if } f \in [-1, 0) \\ \sqrt{1 + K^2(f)} & \text{if } f \in [0, 1) \\ |K(f)| & \text{if } f \in [1, 2), \end{cases}$$

which takes arbitrarily small values in the vicinity of $f = 1.5$, where $K(f)$ vanishes. Hence the bound on $D^-(\Lambda_2)$ is a strict inequality. Another calculation yields (recall that we take $\sigma_{\min}(\boldsymbol{A}) = \infty$ if $\boldsymbol{A} = \boldsymbol{0}$)

$$\sigma_{\min}\big(\boldsymbol{G}_{2,\mathcal{C}_f}(f)\big) = \begin{cases} \infty & \text{if } f \in [-1, 0), \\ 1 & \text{if } f \in [0, 1), \\ 1 & \text{if } f \in [1, 2). \end{cases}$$

Hence, the bound on $D^-(\Lambda_1)$ is not a strict inequality. In summary, we obtain the following necessary conditions on the joint densities:

$$D^-(\Lambda_1, \Lambda_2) \geq 4, \quad D^-(\Lambda_1) \geq 2, \quad \text{and} \quad D^-(\Lambda_2) > 1.5.$$

If Λ_1 and Λ_2 have uniform densities of d_1 and d_2 respectively, the resulting outer bounds on the density regions for stable sampling can be viewed as a region in $\mathbb{R}^2$, as illustrated in Figure 3. Also illustrated in the same figure, are the outer bounds on the density regions for consistent reconstruction (see the Introduction for a definition) for an example with the same parameters [4].

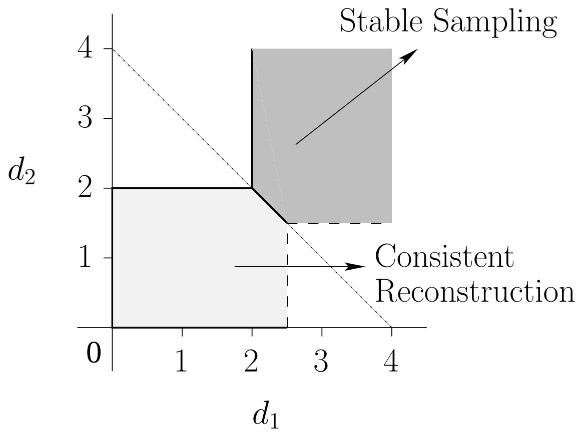

Fig. 3. Density regions for stable sampling and consistent reconstruction.

However, it is not immediately clear whether all densities satisfying the necessary conditions are achievable or how to achieve them.

IV. Periodic Nonuniform Sampling

For the case of periodic nonuniform MIMO sampling, we present conditions that are both necessary and sufficient for stable reconstruction. Although these conditions are not explicitly on the sampling densities, their application often leads to sampling rates achieving minimum densities given by the necessary density conditions.

A. Modeling

We begin by recasting periodic nonuniform MIMO sampling as a uniform sampling problem with a virtual MIMO channel. Consider the case where the p-th channel output $y_p(t)$ is sampled at

$$t \in \Lambda_p = \{nT_p + \lambda_{kp} : k = 0, \ldots, K_p - 1, n \in \mathbb{Z}\}.$$

The period of the sampling pattern for the p-th output channel is T_p, and its uniform sampling density is K_p/T_p. First, consider the case where all the periods are equal, *i.e.*, $T_p = T$. Then, we can decompose Λ_p into a union of K_p uniform sampling sets of density $1/T$, $\Lambda_p = \cup_{k=0}^{K_p-1}(T\mathbb{Z}+\lambda_{kp})$ Consider a virtual MIMO channel whose frequency response matrix is obtained by performing the following modification to $\boldsymbol{G}(f)$ of (6). We replace the p-th row of $\boldsymbol{G}(f)$, namely $\boldsymbol{G}_{p,\bullet}(f)$, by the following K_p rows: $\boldsymbol{G}_{p,\bullet}(f)e^{-j2\pi f\lambda_{kp}}$, $k = 0, \ldots, K_p - 1$. The new channel matrix has $\sum_p K_p$ rows, and the samples of the new outputs taken at $t = nT$ are precisely equal to the samples of the old MIMO channel outputs taken on the periodic nonuniform sampling sets $\{\Lambda_p\}$ and reordered.

Next suppose that the different channels have unequal but *commensurate* sampling periods, *i.e.*, that the ratios of sampling periods are rational numbers: $T_p = (m_p/n_p)T$, for some $m_p, n_p \in \mathbb{N}$, and $T \in \mathbb{R}$. In this case, a common period for all the sampling sets $\{\Lambda_p\}$ is $T\prod n_p$, and an argument as before allows us to convert this to uniform sampling of the outputs of a virtual MIMO channel. This recasting into uniform MIMO sampling applies to most periodic nonuniform sampling schemes, except those with non-commensurate periods. (In the latter case, the entire sampling scheme is not periodic). Of course, the price to pay is that the virtual MIMO channel has many more outputs. Thus, in the sequel, we present results for uniform MIMO sampling only, illustrating the application to periodic nonuniform sampling by examples.

In the discussion that follows, we assume that spectral support $\mathcal{F}_r$ of each of the inputs is a finite union of disjoint intervals. We call this a *multiband structure.* Also, for convenience we index the inputs and outputs by $\mathcal{R} = \{0, ..., R-1\}$ and $\mathcal{P} = \{0, ..., P-1\}$, respectively. The channel outputs are sampled at $t = nT$, $n \in \mathbb{Z}$, producing the vector sequence $\boldsymbol{z}[n] \stackrel{\text{def}}{=} \boldsymbol{y}(nT)$ with Fourier transform $\boldsymbol{Z}[\nu] \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \boldsymbol{z}[n] \exp\{-i2\pi n\nu\}$ given by

$$\boldsymbol{Z}[\nu] = \frac{1}{T}\sum_{l\in\mathbb{Z}} \boldsymbol{G}\Big(\frac{\nu+l}{T}\Big)\boldsymbol{X}\Big(\frac{\nu+l}{T}\Big), \quad \nu \in [0, 1)\ . \tag{13}$$

We model the reconstruction process by

$$\tilde{\boldsymbol{x}}(t) = \sum_{n\in\mathbb{Z}} \boldsymbol{h}(t-nT)\boldsymbol{z}[n],\ \boldsymbol{h}(t) \in \mathbb{C}^{R\times P}\ . \tag{14}$$

This is the most general linear transformation that allows invariance of the entire MIMO system (consisting of the channel, the samplers and the reconstruction block) to a time-shift by a multiple of T. Taking its Fourier transform yields $\tilde{\boldsymbol{X}}(f) = \boldsymbol{H}(f)\boldsymbol{Z}[fT]$, $\quad f \in \mathbb{R}$, where $\boldsymbol{H}(f)$, the Fourier transform of $\boldsymbol{h}(t)$, is the *reconstruction matrix.* Owing to the periodicity of $\boldsymbol{Z}[\nu]$, we can rewrite (13) and the reconstruction relationship compactly as

$$\boldsymbol{Z}[fT] = \boldsymbol{\mathcal{G}}(f)\boldsymbol{\mathcal{X}}(f), \tag{15}$$

$$\tilde{\boldsymbol{\mathcal{X}}}(f) = \boldsymbol{\mathcal{H}}(f)\boldsymbol{Z}[fT], \tag{16}$$

for $f \in [0, 1/T)$, where $\boldsymbol{\mathcal{X}}(f)$ and $\tilde{\boldsymbol{\mathcal{X}}}(f)$ are the *modulated input and reconstructed vectors* whose entries are

$$\boldsymbol{\mathcal{X}}_{Rl+r}(f) = X_r\Big(f + \frac{l}{T}\Big), \quad (r, l) \in \mathcal{R}\times\mathbb{Z}, \tag{17}$$

with a similar definition for $\tilde{\boldsymbol{\mathcal{X}}}_{Rl+r}(f)$, while $\boldsymbol{\mathcal{G}}(f)$ and $\boldsymbol{\mathcal{H}}(f)$ are the *modulated channel and reconstruction matrices* whose entries are, for $(p,r,l) \in \mathcal{P} \times \mathcal{R} \times \mathbb{Z}$,

$$\boldsymbol{\mathcal{G}}_{p,Rl+r}(f) = \frac{1}{T} G_{pr}\Big(f + \frac{l}{T}\Big), \tag{18}$$

$$\boldsymbol{\mathcal{H}}_{Rl+r,p}(f) = H_{rp}\Big(f + \frac{l}{T}\Big). \tag{19}$$

Note that only a finite summation is involved in (15) because the components of $\boldsymbol{X}(f)$ are bandlimited implying that only a finite number of entries in $\boldsymbol{\mathcal{X}}(f)$ are nonzero.

B. Stable Sampling

Define the following two *spectral index sets* at frequency $f \in [0, 1/T)$:

$$\begin{aligned} \mathcal{K}_f^{\circ} &= \Big\{(r,l) \in \mathcal{R} \times \mathbb{Z} : \Big(f + \frac{l}{T}\Big) \in \mathcal{F}_r\Big\}, \\ \mathcal{K}_f &= \Big\{Rl + r : (r,l) \in \mathcal{R} \times \mathbb{Z} \text{ and } \Big(f + \frac{l}{T}\Big) \in \mathcal{F}_r\Big\}. \end{aligned} \tag{20}$$

Let $\mathcal{K}_f^c = \mathbb{Z} \backslash \mathcal{K}_f$ denote the complement of $\mathcal{K}_f$. We now have the following proposition, easily demonstrated by using an argument similar to the one in [10].

Proposition 1: Suppose that sets $\mathcal{F}_r$, $r \in \mathcal{R}$ have multiband structure. Then $\mathcal{K}_f$ is piecewise constant on $[0, 1/T)$, *i.e.*, there exists a collection of disjoint intervals $\mathcal{I}_m$ of the form $[\alpha, \beta)$, and sets $\mathcal{K}_m$, $m = 1, \dots, M$ such that $\mathcal{K}_f = \mathcal{K}_m$, for $f \in \mathcal{I}_m$, and $\cup_{m=1}^M \mathcal{I}_m = [0, 1/T)$.

Conditions for stability of uniform MIMO sampling are established by determining the frame bounds in Definition 2.

Theorem 3: The best frame bounds for the MIMO sampling problem are given by

$$A = T \operatorname*{ess\,inf}_{f \in [0,1/T]} \lambda_{\min}\big(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_f}(f) \boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\big), \tag{21}$$

$$B = T \operatorname*{ess\,sup}_{f \in [0,1/T]} \lambda_{\max}\big(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_f}(f) \boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\big). \tag{22}$$

In particular, $A > 0$ and $B < \infty$ are necessary and sufficient conditions for stable reconstruction of the MIMO inputs.

The following corollary to Theorem 3 provides a simpler sufficient condition for the stability of the MIMO sampling scheme.

Corollary 2: Suppose that $\boldsymbol{G}(f)$ is such that $G_{pr}(f)$ is continuous for $f \in \overline{\mathcal{F}}_r$, and $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ has full column rank for all $m \in \mathcal{M}$, $f \in \overline{\mathcal{I}}_m = [\gamma_m, \gamma_{m+1}]$. Then the MIMO sampling scheme is stable.

We illustrate the MIMO sampling result of Theorem 3 for a simple MIMO channel.

Example 2: Consider a MIMO channel with $R = 2$ inputs and $P = 4$ outputs having the following frequency response matrix:

$$\boldsymbol{G}(f) = \begin{pmatrix} 1 & 1 \\ 1 & 1 + e^{-j2\pi f} \\ e^{-j2\pi f} & 0.25 + e^{-j4\pi f} \\ 1 + 0.5e^{-j2\pi f} & 1 + e^{-j4\pi f} \end{pmatrix}.$$

Let the input spectra $X_0(f)$ and $X_1(f)$ have supports

$$\mathcal{F}_0 = [0, 0.4) \cup [0.75, 0.9) \quad \text{and} \quad \mathcal{F}_1 = [0.25, 0.5).$$

Each output is a multiband signal supported on $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 = [0, 0.5) \cup [0.75, 0.9)$. A naïve way to reconstruct the inputs is to first reconstruct the individual outputs and then invert the channel. This method requires a minimum sampling rate of $\mu(\mathcal{F}) = 0.65$ for each channel output. However, we demonstrate in this example that we can jointly reconstruct the inputs from fewer samples.

Let the sampling period be $T = 4$. It is then easy to verify that $\mathcal{I}_1 = [0, 0.15)$ and $\mathcal{I}_2 = [0.15, 0.25)$. Furthermore, (20) and Proposition 1 imply that

$$\mathcal{K}_f^{\circ} = \begin{cases} \{(0,0),(0,1),(0,3),(1,1)\}, & \text{if } f \in \mathcal{I}_1, \\ \{(0,0),(1,1)\}, & \text{if } f \in \mathcal{I}_2. \end{cases}$$

Therefore $\mathcal{K}^1 = \{0, 2, 6, 3\}$ and $\mathcal{K}^2 = \{0, 3\}$. A simple calculation determines $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f), m = 1, 2$ and it can be verified numerically that

$$\begin{aligned} \operatorname{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^1}(f)\big) &= 4, \quad \forall f \in \overline{\mathcal{I}_1}, \\ \operatorname{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^2}(f)\big) &= 2, \quad \forall f \in \overline{\mathcal{I}_2}. \end{aligned}$$

Since $\boldsymbol{G}(f)$ is continuous, we conclude using Corollary 2, that stable perfect reconstruction of the inputs is possible from the channel output samples. Hence, it suffices to sample each output at a rate $1/T = 0.25$ for perfect stable reconstruction of the channel inputs, instead of the sampling them at a rate $\mu(\mathcal{F}) = 0.65$ which is required for the naïve approach. Finally, note that the total combined sampling density of the outputs is $P/T = 1$, while the minimum density, as dictated by Theorem refthm:sampling.density.result is $\mu(\mathcal{F}_0) + \mu(\mathcal{F}_1) = 0.8$.

In Example 2, we showed that the combined sampling density of 1 is achievable, but the lower bound on this density is 0.8. In the next example we find a nonuniform MIMO sampling scheme that closes the gap.

Example 3: Let the inputs signal characteristics and the channel transfer function matrix be the same as in Example 2. Let the channel outputs be sampled on the sets $\Lambda_p = \{20n + \lambda_{kp} : k = 0, \dots, K_p - 1\}$, where $(K_1, K_2, K_3, K_4) = (0, 3, 5, 8)$ and

$$\{\lambda_{kp} : 0 \le k < K_p\} = \begin{cases} \emptyset & p = 0, \\ \{1, 8, 14\} & p = 1 \\ \{2, 5, 8, 13, 18\} & p = 2 \\ \{0, 2, 4, 5, 7, 8, 14, 17\} & p = 3 \end{cases}$$

These are all periodic nonuniform sampling sets with common period of $T = 20$, and consisting of 16 cosets in all. Hence, the virtual MIMO channel has a 16×2 frequency response matrix $\tilde{\boldsymbol{G}}(f)$. Since the band edges of $\mathcal{F}_0$ and $\mathcal{F}_1$ are all multiples of 0.05, we trivially obtain $M = 1$, $\mathcal{I}_1 = [0, 0.05)$, and

$$\mathcal{K}^1 = \{0, 2, 4, 6, 8, 10, 12, 14, 30, 32, 34\} \cup \{11, 13, 15, 17, 19\}.$$

Now, $\tilde{\boldsymbol{\mathcal{G}}}_{\bullet,\mathcal{K}^1}(f)$ is a continuous 16×16 matrix, whose rank is verifiable to be 16 for all f. By Corollary 2, we conclude that stable and perfect reconstruction of the channel inputs is possible from these periodic nonuniform MIMO samples. In fact, the stability bounds are $A = 8.0724 \times 10^{-4}$ and $B = 3.6833$. The sampling density of Λ_p is $d_p = K_p/T$, so that

$$(d_0, d_1, d_2, d_3) = (0, 0.15, 0.25, 0.5)$$

is an achievable point in density region for stable sampling. Obviously, the densities (d_0, d_1, d_2, d_3) must meet all the necessary conditions for stable sampling in Theorem 1. In particular, the total combined sampling rate of all the outputs is $16/T = 0.8$, which is precisely equal to the minimum joint sampling density required, namely $\mu(\mathcal{F}_0) + \mu(\mathcal{F}_1)$. Finally, we learn from this example that we need not sample the different outputs at the same rate. In fact, one of the channels is not sampled at all, unlike in Example 2 where, due to uniform sampling, we required samples from all channel outputs.

C. Existence of Continuous Reconstruction Filters

In practice, one would reconstruct only a version of the set of inputs that is uniformly sampled at a sufficiently high rate, and implement $\boldsymbol{H}(f)$ using finite impulse response (FIR) digital filters. The continuous-time version could then be reconstructed by a bank of conventional D/A converters on the reconstructed discrete-time signals. In particular, it is desirable to use a reconstruction filter matrix $\boldsymbol{H}(f)$ that is continuous in f. Then $\boldsymbol{H}(f)$ can be approximated arbitrarily closely in the $\mathcal{H}^\infty$ sense (and thus ensure an arbitrarily small worst-case $\mathcal{L}^2$ reconstruction error) by choosing sufficiently long FIR filters [6]. The existence of a continuous $\boldsymbol{H}(f)$ is guaranteed under a stronger set of conditions than in Theorem 3.

Theorem 4: Suppose that the MIMO frequency response matrix $\boldsymbol{G}(f)$ is such that $G_{pr}(f)$ is continuous for $f \in \overline{\mathcal{F}}_r$. Then there exists a reconstruction filter matrix $\boldsymbol{H}(f)$ continuous in f, that achieves stable reconstruction of the MIMO channel inputs if and only if

$$\operatorname{rank}\left(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)\right) = |\mathcal{K}^m|, \quad \forall f \in \operatorname{int}\mathcal{I}_m = (\gamma_m, \gamma_{m+1}), \tag{23}$$

$$\operatorname{rank}\left(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{J}_m}(\gamma_m)\right) = |\mathcal{J}_m|, \quad m \in \mathcal{M}. \tag{24}$$

$$\text{where} \quad \begin{aligned} \mathcal{J}_m &= \mathcal{K}^m \cup \mathcal{K}^{m-1}, \quad m = 2, \ldots, M, \\ \mathcal{J}_1 &= \mathcal{K}^1 \cup (\mathcal{K}^M \oplus R). \end{aligned} \tag{25}$$

Remark 1: A simple necessary condition for perfect reconstruction using continuous reconstruction filters is that $P \geq \max_m |\mathcal{J}_m|$.

Example 4: Continuing Example 2, let $R = 2$, $T = 4$ Then the index sets defined in (25) are $\mathcal{J}_1 = \mathcal{K}^1 \cup (\mathcal{K}^2 \oplus 2) = \{0, 2, 6, 3, 5\}$, and $\mathcal{J}_2 = \mathcal{K}^2 \cup \mathcal{K}^1 = \{0, 2, 3, 6\}$.Hence, $P \geq \max_m |\mathcal{J}_m| = 5$ is necessary for the existence of a continuous $\boldsymbol{H}(f)$, and clearly, the $\boldsymbol{G}(f)$ of Example 2 does not suffice. So let us append a new row, $[0.25 + e^{-j4\pi f}, e^{-j2\pi f}]$, beneath the last row of $\boldsymbol{G}(f)$, thereby making the MIMO channel a two-input five-output channel. The rank condition in (23) holds because the matrix $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$ of Example 2 has full column rank, and adding an extra row to $\boldsymbol{G}(f)$ (and hence to $\boldsymbol{\mathcal{G}}(f)$ also) does not lower the column rank of $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}^m}(f)$. A numerical calculation yields the frame bounds for the MIMO sampling scheme: $A = \operatorname{ess\,inf}_{f \in [0, \frac{1}{T}]} \lambda_{\min}(T\boldsymbol{S}(f)) = 0.1251$, $B = \operatorname{ess\,sup}_{f \in [0, \frac{1}{T}]} \lambda_{\max}(T\boldsymbol{S}(f)) = 1.1105$. The other rank condition in (24), which needs to be verified at cell boundaries, also holds. Now, Theorem 4 guarantees the existence of a continuous filter matrix $\boldsymbol{H}(f)$ that achieves stable reconstruction of the MIMO channel inputs.

D. Optimal Reconstruction

When the conditions for its existence are satisfied, the continuous reconstruction filter matrix $\boldsymbol{H}(f)$ is in general nonunique, and a particular solution can be selected using additional criteria (for examples of such designs in the single channel case, see [11]). Necessary and sufficient conditions for the existence of FIR perfect reconstruction filters when the channel itself is FIR are presented in [6]. These conditions depend on the channel, input multiband structures, and sampling rate, and generalize previous results on multichannel deconvolution and filter banks. However, in general, perfect reconstruction FIR filters do not exist for the MIMO sampling problem. Formulating the reconstruction filter design problem as a minimax optimization, it can be recast as a standard semi-infinite linear program admitting efficient numerical solutions [6]. The generality of the MIMO setting allows this algorithm to be used for various other sampling schemes that fit into the MIMO framework as special cases.

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