Necessary Density Conditions for MIMO Sampling of Multiband Inputs[†]

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Abstract — **We consider a sampling scheme where a set of multiband input signals is passed through a MIMO channel and the outputs are sampled nonuniformly. MIMO sampling is a general scheme that encompasses various other schemes, including Papoulis' generalized sampling and nonuniform sampling as special cases. We present necessary density conditions for stable nonuniform MIMO sampling and consistent reconstruction of the multiband inputs.**

I. PROBLEM DESCRIPTION

Let a set of multiband signals $x_r(t)$, $r = 1, \ldots, R$ with *spectral supports* $\mathcal{F}_r \subseteq \mathbb{R}$ be passed through a MIMO channel consisting of linear time-invariant filters to produce outputs $y_p(t)$, $p = 1, \ldots, P$ given by $y_p(t) = \sum_{r=1}^{R} g_{pr} * x_r(t)$ where $*$ denotes convolution and $g_{pr}(t) \in L^2(\mathbb{R})$ are the channel filter impulse responses. The outputs $y_p(t)$ are subsequently sampled on nonuniform sets $\Lambda_p = \{\lambda_{np} : n \in \mathbb{Z}\}$. This general scheme, called MIMO sampling, subsumes various other schemes including Papoulis' generalized sampling [1] as special cases. We address the following questions: (a) what are necessary conditions on $\{\Lambda_p\}$ for stable reconstruction of the inputs $x_r(t)$ from the MIMO output samples $\{y_p(\lambda_{np})\}$? (b) what are necessary conditions on $\{\Lambda_p\}$ for *consistent reconstruction*, i.e., $\exists x_r(t)$ such that $y_p(\lambda_{np}) = c_{np}$ for any sequence $\{c_{np} : n \in \mathbb{Z}, p = 1, \ldots, P\} \in l^2$. Our results, expressed in terms of sampling densities, use ideas from [2] to generalize Landau's classical results [3].

II. NECESSARY DENSITY CONDITIONS

Let $\mathcal{B}(\mathcal{F}) = \{x(t) \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : X(f) = 0, \forall f \notin \mathcal{F}\}$, where $X(f)$ is the Fourier transform of $x(t)$. Let $m(\cdot)$ denote the Lebesgue measure. Let $\emptyset$ denote the empty set and $\mathcal{S}^c$ the complement of a set $\mathcal{S}$. For any matrix $\boldsymbol{A}$, let $\boldsymbol{A}^H$ denote its conjugate transpose and $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$, the submatrix of $\boldsymbol{A}$ whose rows and columns are indexed by sets $\mathcal{R}$ and $\mathcal{C}$ respectively. Let $\sigma_{\min}(\boldsymbol{A})$ and $\sigma_{\max}(\boldsymbol{A})$ denote the smallest and largest *nonzero* singular values of $\boldsymbol{A}$. If $\boldsymbol{A} = 0$, we take $\sigma_{\min}(\boldsymbol{A}) = \infty$. Let $\mathcal{R} = \{1, \ldots, R\}$ and $\mathcal{P} = \{1, \ldots, P\}$ denote index sets for the input and output components.

Writing the MIMO channel input and output signals in vector form as $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$, we have $\boldsymbol{y}(t) = \boldsymbol{g}(t) * \boldsymbol{x}(t)$ or $\boldsymbol{Y}(f) = \boldsymbol{G}(f)\boldsymbol{X}(f)$ where $\boldsymbol{G}(f)$ is the channel transfer function. The space of inputs is $\mathcal{H} \overset{\text{def}}{=} \mathcal{B}(\mathcal{F}_1) \times \cdots \times \mathcal{B}(\mathcal{F}_R)$.

Let $\Lambda_p = \{\lambda_{np} : n \in \mathbb{Z}\}$, $p \in \mathcal{P}$ be a collection of sampling sets. Then $\{\Lambda_p\}$ is a *stable collection of MIMO sampling* w.r.t. $\boldsymbol{G}(f)$ for the space $\mathcal{H}$ if $\exists A, B > 0$ such that

$A\|\boldsymbol{x}\|^2 \leq \sum_{p=1}^{P} \sum_n |y_p(\lambda_{np})|^2 \leq B\|\boldsymbol{x}\|^2$, $\forall \boldsymbol{x} \in \mathcal{H}$, where $\boldsymbol{Y}(f) = \boldsymbol{G}(f)\boldsymbol{X}(f)$. Dually, $\{\Lambda_p\}$ is a *collection of consistent reconstruction* w.r.t. $\boldsymbol{G}(f)$ for $\mathcal{H}$ if, $\forall \{c_{np}\} \in l^2$, $\exists \boldsymbol{x} \in \mathcal{H}$ and $\boldsymbol{Y}(f) = \boldsymbol{G}(f)\boldsymbol{X}(f)$ such that $y_p(\lambda_{np}) = c_{np}$. These definitions generalize those for simple multiband sampling [4].

Let $\text{ext}^{\pm}$ denote "sup" and "inf" respectively and $\#(\cdot)$, the cardinality of a set. Then the *joint upper and lower densities* of a collection of discrete sets Λ_p, $p \in \mathcal{P}$ are defined as

$$D^{\pm}(\Lambda_1, \ldots, \Lambda_P) = \lim_{\gamma \to \infty} \text{ext}^{\pm} \nu_\gamma^{\pm}(\Lambda_1, \ldots, \Lambda_P)/(2\gamma),$$

where $\nu_\gamma^{\pm}(\Lambda_1, \ldots, \Lambda_P) = \text{ext}^{\pm}_\tau \sum_{p=1}^{P} \#(\Lambda_p \cap [\tau - \gamma, \tau + \gamma])$ are the maximum and minimum number of points of the collection $\{\Lambda_p : p = 1, \ldots, P\}$ contained in any interval of length 2γ.

Theorem 1 *For stable MIMO sampling, we require*

$$D^{-}(\{\Lambda_p : p \in \Pi\}) \geq \sum_{r \in \mathcal{R}} m(\mathcal{F}_r) - \int \text{rank}\, \boldsymbol{G}_{\Pi^c, \mathcal{C}_f}(f) df \quad (1)$$

$\forall \Pi \subseteq \mathcal{P}$, *where* $\mathcal{C}_f = \{r : f \in \mathcal{F}_r\}$. *Further, (1) is a strict inequality if* $\text{ess inf}_{f \in \mathcal{F}} \sigma_{\min}(\boldsymbol{G}_{\Pi^c, \mathcal{C}_f}(f)) = 0$, $\mathcal{F} = \bigcup_{r \in \mathcal{R}} \mathcal{F}_r$ *for some* $\Pi \neq \mathcal{P}$. *In addition* $\text{ess sup}_{f \in \mathcal{F}} \sigma_{\max}(\boldsymbol{G}_{\Pi^+, \mathcal{C}_f}(f)) < \infty$, *where* $\Pi^+ = \{p \in \mathcal{P} : D^{+}(\Lambda_p) > 0\}$.

Theorem 2 *For consistent MIMO reconstruction, we require*

$$D^{+}(\{\Lambda_p : p \in \Pi\}) \leq \int \text{rank}\, \boldsymbol{G}_{\Pi, \mathcal{C}_f}(f) df \quad (2)$$

$\forall \Pi \subseteq \mathcal{P}$, *where* $\mathcal{C}_f = \{r : f \in \mathcal{F}_r\}$. *Further, (2) is a strict inequality if* $\text{ess inf}_{f \in \mathcal{F}} \sigma_{\min}(\boldsymbol{G}_{\Pi, \mathcal{C}_f}(f)) = 0$, $\mathcal{F} = \bigcup_{r \in \mathcal{R}} \mathcal{F}_r$ *for some* $\Pi \neq \emptyset$.

Theorems 1 and 2 generalize Landau's density results for stable sampling and interpolation respectively. Letting $\Pi = \mathcal{P}$ in (1), we obtain $D^{-}(\Lambda_1, \ldots, \Lambda_P) \geq \sum_{r=1}^{R} m(\mathcal{F}_r)$, i.e., the total output sampling density cannot be less than the combined bandwidth of the inputs. Similarly, for $\Pi = \mathcal{P}$ (2) produces $D^{+}(\Lambda_1, \ldots, \Lambda_P) \leq \sum_{r=1}^{R} m(\mathcal{F}_r)$, i.e., the combined density of the sampling sets cannot exceed the combined bandwidth of the input signals. In fact, each theorem provides a total of $2^P - 1$ easily computable conditions—one for the joint upper or lower density of each sub-collection of $\{\Lambda_p, p \in \mathcal{P}\}$.

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